Methodus Differentialis:

SIVE

TRACTATUS

DE

SUMMATIONE

ET

INTERPOLATIONE

SERIERUM INFINITARUM.

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LONDINI:

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M DCC LXIV.

PRÆFATIO.

CUM *sæpissime eveniat ut Series adeo lente convergant ad veritatem, ut non magis conducant ad finem propositum quam si revera essent divergentes; exhibui quædam Theoremata in parte priore hujus Tractatus, quibus prompte acceditur ad valores earum Serierum quæ omnium lentissime approximant: Et eo quidem consilio, ut Problemata quæ dependent ex Quadraturis, eodem jure pro solutis haberi possint quo ea quæ reducuntur ad Æquationes affectas. Hic enim non inquiro quænam Series sint summabiles (ut ex titulo nonnulli forsan conjiciant) sed quibus artibus assequamur valores earum quæ summari nequeunt.*

Hujus generis Tractatum edidit Jacobus Gregorius *in Appendice ad* Veram Circuli & Hyperbolæ Quadraturam, *ubi docuit modum facillimum approximandi ad Areas harum Curvarum ex datis perpaucis polygonis pauciorum laterum. Eoque pacto methodum Exhaustionum*

 Archimedis

Archimedis *expeditam adeo reddidit, ut ſi eadem methodus pari cum ſucceſſu ad reliquas Curvas extendi poſſet, fruſtra impenderetur ulterior opera in eliciendis Areis. Quodque* Gregorius *ibi effecit in Serie polygonorum, idem fere nos hic generaliter proponimus in qualibet aliâ quæ ſimplici gaudet Terminorum relatione.*

Newtonum *hâc de re olim meditatum fuiſſe conſtat ex Epiſtolâ ejus priori ad* Oldenburgum *quæ impreſſa habetur in* Commercio Epiſtolico Collinii. *Nimirum poſtquam computo produxerat Aream Circuli ad ſedecim decimales, inquit,* Si alias artes adhibuiſſem, potui per eundem numerum Terminorum Seriei, perveniſſe ad multo plura loca figurarum, puta viginti quinque aut amplius: Sed animus fuit hic oſtendere quid per ſimplex Seriei computum præſtari poſſet. *Attamen inter ſcripta ejus (hactenus ſaltem edita) ne veſtigium quidem extat unde conjecturam faciamus de hiſce artibus, etiamſi inſignem eas proferendi occaſionem nactus eſſet in hâc ipſiſſimâ Epiſtolâ. In priori etiam Epiſtolâ ad eundem, ibidem impreſſâ; narrat ſe excogitaſſe quædam circa Reductionem infinitarum Serierum in finitas ubi rei natura tulerit: Quæ ſi inter opera ejus poſthuma ſuperſint, proculdubio huic doctrinæ lucem haud exiguam præbebunt: Nam Theoremata generalia quæ valores Serierum exhibent accurate, quando id fieri poteſt, in aliis caſibus neceſſe approximabunt, modo rite adhibeantur.*

Principium quod in hunc effectum vulgo adducitur, eſt Aſſumptio differentiæ duorum valorum ſucceſſivorum alicujus quantitatis, ut exinde formentur Termini quorum ſumma prius nota fuerat; idem ſcilicet principium, quo olim uſus eſt Newtonus *ad Ordinatam Curvæ deducendam*

ex

ex data Area. Quod licet ſit univerſale in Quadraturis, eſt particulare tantummodo in Summationibus; quippe ad eas ſolummodo Series applicabile quarum Termini poſſunt aſſignari: quum tamen æque facilis ſit aſſignatio Summæ atque Termini in iis Seriebus quæ magnâ ex parte in Quadraturis prodire ſolent.

Fundamentum longe generalius ſubminiſtravit Methodus Newtoni Differentialis: *Ille utique deſcribit Curvam Parabolicam per extremitates quotlibet Ordinatarum ſive Terminorum; eâque ratione aſſignat valorem cujuſvis intermedii per Seriem infinitam; quæ tamen ad veritatem non accedet ſi Terminus ille longe diſtet ab initio. Ut obtinerem igitur Terminos Serierum remotiſſimos, deſcripſi figuram Hyperbolicam per extremitates Terminorum; & res ſucceſſit, prodeunte valore Termini quantumvis diſtantis, per Seriem convergentem. Sed & hoc Problemate generaliter ſoluto, non latuit caſus ejus facillimus, utpote inventio Termini infinito intervallo diſtantis a principio; quæ quidem æquipollet Summationi Serierum. Cæterum deſcriptio Curvæ cujuſcunque Geometricæ per data puncta, uni tantum Serierum generi ſufficit: atque alia ſunt innumera nullatenus tractanda ex hoc fundamento. Nam valor Termini per Parabolam aut Hyperbolam inventus, non approximat niſi ubi differentiæ ſecundum præſcripta* Newtoniana *ſumptæ conſtituant progreſſionem celeriter ſatis decreſcentem.*

Hiſce perſpectis, me tandem contuli ad contemplandam Relationem Terminorum, proprietatem Serierum maxime inſignem & ſimplicem, quæ vulgo ad eas continuandas adhiberi ſolet: Non enim ignorabam Moivræum *hanc Terminorum proprietatem ſummo cum ſucceſſu in Algebram*

bram introduxisse, tanquam fundamentum ad difficillima circa Series recurrentes Problemata solvenda: Ideoque decreveram experiri num ad alias quoque extendi posset: Quod sane dubitabam cum tanta sit inter recurrentes atque alias Series discrepantia. At facto experimento, res præter spem successit, deprehendi enim Inventum hocce Moivræanum *continere principia generalissima & etiam simplicissima non solum in Seriebus recurrentibus, sed & in quibuslibet aliis, in quibus relatio Terminorum variatur secundum legem quamcunque regularem. Nam Terminorum relatio, etiamsi sit variabilis, facile tamen assignabilis est: Atque exinde Summationes & Interpolationes, aliaque ejusmodi Problemata difficiliora perducuntur ad quoddam genus Æquationum, quæ præter Radicem extrahendam involvunt alias quantitates incognitas quæ nequeunt eliminari; quo non obstante, harum Æquationum resolutio perficitur nonnunquam summâ facilitate, nonnunquam verò non succedit nisi mediantibus* Moivræi *Inventis circa assignationem Terminorum in Seriebus recurrentibus. Et de hoc argumento totus fere agit hicce Libellus.*

Problema de invenienda Unciâ mediâ in permagnâ dignitate binomii solutum erat a Moivræo *ante aliquot annos quam ego idem attigeram: Nec probabile est quod in hunc usque diem de eodem cogitassem, ni suggessisset Spectatissimus Vir, Dr. Alex. Cuming, se plurimum suspicari an idem solvi posset per* Methodum Differentialem Newtoni.

TRACTATUS

TRACTATUS
DE
SUMMATIONE & INTERPOLATIONE
Serierum Infinitarum.

INTRODUCTIO.

SICUT Curvæ non determinantur ex datis quotlibet earum ordinatis, ſed ex relatione inter Abſciſſas & Ordinatas in genere; ita neque Series determinantur ex datis quotlibet earum Terminis, ſed ex relatione inter Terminos ſucceſſivos. Quantitates enim quotcunque, numero tamen finitæ, conſtituere poſſunt Terminos in diverſis Seriebus: unica vero eſt Series in qua iidem ſunt Termini initiales, eademque lex formandi reliquos in infinitum. Relationes igitur Terminorum primo loco indagandæ ſunt; deinde hæ inventæ denotandæ ſunt Æquationibus Differentialibus perinde ac *Carteſius* definivit Curvas Æquationibus Algebraicis:

braicis: quibus habitis Problemata circa Summationem & Interpolationem, aliaque iſtiuſmodi ad Series ſpectantia ſolventur per Analyſin non minus certam quam eſt Algebra vulgaris.

De Relatione Terminorum.

Termini Serierum binatim, ternatim, aut plures numero ſumpti relationem quandam magna ex parte inter ſe obtinent ſimplicem & obviam, per quam Series determinantur & producuntur ad libitum. Ut ſi dividatur unitas per $1-x$, prodibit progreſſio Geometrica in quâ Terminus quivis conſequens eſt ad illum immediate antecedentem ut x ad unitatem. Per hanc enim Proprietatem Series $1+x+x^2+x^3+x^4+x^5+$ &c. ab alia quavis diſtinguitur & producitur in infinitum.

Proponatur fractio $\frac{1}{r+sx+tx^2}$ in Seriem reſolvenda: illum in finem finge $y=\frac{1}{r+sx+txx}$; ac ducendo utramque partem in Denominatorem, habebitur $y\times\overline{r+sx+txx}=1$, vel $y\times\overline{r+sx+txx}-1=0$. Pro y ſubſtitue Seriem debitæ formæ $A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+$ &c. atque reſultabit

$$\left.\begin{matrix}rA\\-1\end{matrix}\right.\left.\begin{matrix}+rB\\+sA\end{matrix}\right\}x\left.\begin{matrix}+rC\\+sB\\+tA\end{matrix}\right\}x^2\left.\begin{matrix}+rD\\+sC\\+tB\end{matrix}\right\}x^3\left.\begin{matrix}+rE\\+sD\\+tC\end{matrix}\right\}x^4\left.\begin{matrix}+rF\\+sE\\+tD\end{matrix}\right\}x^5+\&c.=0.$$

Ubi ponendo membra homologa nihilo æqualia, ad determinandum Coefficientes aſſumptos, erit $rA-1=0$, $rB+sA=0$; deinde $rC+sB+tA=0$, $rD+sC+tB=0$, $rE+sD+tC=0$, & ſic in infinitum. Ex quibus conſtat eandem ubique prodire relationem inter tres quoſcunque Terminos ſucceſſivos. Similiter uſurpando pro y Seriem hujus formæ $Ax^{-2}+Bx^{-3}+Cx^{-4}+Dx^{-5}+$ &c. & ſubſtituendo eandem in Æquatione, reſultabit

$$\left.\begin{matrix}tA\\-1\end{matrix}\right.\left.\begin{matrix}+tB\\+sA\end{matrix}\right\}x^{-1}\left.\begin{matrix}+tC\\+sB\\+rA\end{matrix}\right\}x^{-2}\left.\begin{matrix}+tD\\+sC\\+rB\end{matrix}\right\}x^{-3}\left.\begin{matrix}+tE\\+sD\\+rC\end{matrix}\right\}x^{-4}+\&c.=0.$$

Atque jam ad determinandos Coefficientes, habemus Æquationes $tA-1=0$, $tB+sA=0$; tum $rA+sB+tC=0$, $rB+sC+tD=0$, $rC+sD+tE=0$, & ita deinceps in reliquis. Ubi patet eaſdem prodire relationes ac in computo ſuperiore, ſumptis tamen Coefficientibus in ordine inverſo. Et quantitates r, s, t, quæ indicant relationem Terminorum, eædem ſunt quæ in Denominatore fractionis.

Hanc

2, 3, &c. pro z. Atque ad eundem modum Series $x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+$ &c. denotatur per hanc $T=\frac{x^z}{zz}$. Tales Æquationes reduci ſemper poſſunt ad eas alterius generis: nam ubi Termini ſunt aſſignabiles, relationes eorum erunt etiam aſſignabiles. Et differentia inter has & illas raro tanta erit quin cuique tuto procedere licuerit prout ſibi viſum fuerit.

Ex hactenus dictis conſtat relationes Terminorum conſequentium derivari ex iis antecedentium, ſcribendo pro z ejus valorem ſucceſſivum $z+1$ in Æquatione Differentiali. Proponatur Æquatio $T'=\frac{z+n}{z}T$; ſcribe $z+1$ pro z, T' pro T, & T'' pro T'; atque orietur $T''=\frac{z+n+1}{z+1}T'$, quæ eſt relatio inter Terminos T' & T''. In hac ultima ſcribe valores variabilium conſequentes $z+1$, T'', & T''', pro antecedentibus z, T' & T'', & obtinebis $T'''=\frac{z+n+2}{z+2}T''$ relationem utique inter T'' & T'''.

Sed & contraria operatione regredi licet ad relationem Terminorum antecedentium ex data ea conſequentium. Sit Æquatio $T''=\frac{z^2-1}{z^3+3z^2+3z+2}T'$, & in eadem ſcribe T pro T', T' pro T'', & $z-1$ pro z; atque habebis $T'=\frac{zz-2z}{z^3+1}T$. Hoc modo regrediendo & progrediendo, poſſunt Series continuari hinc inde in infinitum ubi earum natura tulerit: atque etiamſi ignoretur quinam Termini denotentur per T, T', T'', &c. inſtituere licet ſuper iis computum, tanquam eſſent prorſus cogniti.

Æquationes de quibus hactenus egimus, involvunt duos tantum Terminos Seriei; poſſunt vero involvere plures, & tam Termini quam indeterminata z eſſe plurium dimenſionum. Ceterùm in hoc ſpecimine ſimpliciores tantum attingo.

De Forma & Reductione Serierum.

Poſtquam perduximus Series ad Æquationes Differentiales, monſtrandum eſt qua ratione eædem ſint reſolvendæ in numeris. Nam Analyſtæ munus eſt quantitates quocunque modo determinatas eruere accurate vel quam proxime. Radices autem Æquationum Differentialium commodiſſime reſolvuntur in Series formarum ſequentium

$$A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

$$A + \frac{B}{z} + \frac{C}{z.\overline{z+1}} + \frac{D}{z.\overline{z+1}.\overline{z+2}} + \frac{E}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Quippe ubi z eſt quantitas parva, prior forma erit adhibenda; & poſterior ubi magna. Et hæ Series quæ componuntur ex Factoribus in progreſſione Arithmetica, longe magis idoneæ ſunt huic negotio quam vulgares quæ conflantur ex dignitatibus indeterminatæ aſcendentibus vel deſcendentibus. Inſuper forma poſterior hoc habet commodi, quod in eadem poteſt eſſe z quantumvis fere magna, id quod efficit Seriem celerrime convergentem.

Sin vero harum Serierum formæ per quaſlibet operationes mutentur, debent revocari ad eaſdem quarum priùs erant, ut Termini reddantur homologi eorumque collatio inſtitui poſſit prout res exigit. Ut ſi habeatur Æquatio ſequens;

$$T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Eadem ducta in z priſtinam formam amittet, atque novam induet, proveniente ſcilicet

$$Tz = Az + Bz^2 + Cz^2.\overline{z-1} + Dz^2.\overline{z-1}.\overline{z-2} + Ez^2.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Unde conſtat Terminos ejus comparari non poſſe cum correſpondentibus in Serie priore. Itaque ut debita forma reſtituatur, ſic operor

$$\begin{array}{lll}
Az & \text{- - - - - - - -} = & Az, \\
Bz^2 & \text{- - - - - - -} = & Bz + Bz\,\overline{z-1}, \\
Cz^2.\overline{z-1} & \text{- - - - -} = & * \; 2Cz\,\overline{z-1} + Cz\,\overline{z-1}.\overline{z-2}, \\
Dz^2.\overline{z-1}.\overline{z-2} & \text{- -} = & * \quad * \quad 3Dz.\overline{z-1}.\overline{z-2} + Dz.\overline{z-1}.\overline{z-2}.\overline{z-3} \\
Ez^2.\overline{z-1}.\overline{z-2}.\overline{z-3} & = & * \quad * \text{ - - - - } * \text{ - - - } 4Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.
\end{array}$$

Adeoque colligendo Terminos homologos in unum, Series ad priſtinam formam reducta erit

$$Tz = \left.\begin{array}{l} +A \\ +B \end{array}\right\} z \left.\begin{array}{l} +B \\ +2C \end{array}\right\} z.\overline{z-1} \left.\begin{array}{l} +C \\ +3D \end{array}\right\} z.\overline{z-1}.\overline{z-2} \left.\begin{array}{l} +D \\ +4E \end{array}\right\} z.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Nimirum cum Terminorum *ὁμολογία* neutiquam pendeat ex Coefficientibus A, B, C, D, &c. ſed omnino ex indeterminata z, Terminus $\left.\begin{array}{l} +A \\ +B \end{array}\right\} z$ primus in hac Serie comparari poteſt cum Bz ſecundo in altera, item ſecundus in hac cum tertio in illa, & ſic in reliquis.

Similiter ſi in Æquatione priore $T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + \&c.$ ſcribantur variabilium valores ſuccedentes pro præſentibus, hoc eſt T' pro T, & $z+1$ pro z, emerget

$T' =$

$T' = A + B\overline{z+1} + C\overline{z+1}.z + D\overline{z+1}.z.\overline{z-1} + E\overline{z+1}.z.\overline{z-1}.\overline{z-2} + \&c.$

Est vero

$A \text{ - - - - - - - - - } = A,$

$B\overline{z+1} \text{ - - - - - - } = B + Bz,$

$C\overline{z+1}.z \text{ - - - - } = * \quad 2Cz + Cz.\overline{z-1},$

$D\overline{z+1}.z.\overline{z-1} \text{ - - } = * \text{ - - } * \text{ - - } 3Dz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2}$

$E\overline{z+1}.z.\overline{z-1}.\overline{z-2} = * \text{ - - } * \text{ - - - } * \text{ - - - - - } 4Ez.\overline{z-1}.\overline{z-2} + \&c.$

Et inde

$$T' = \left.\begin{matrix} +A+B \\ +B+2C \end{matrix}\right\} \left.\begin{matrix} +C \\ +3D \end{matrix}\right\} z \left.\begin{matrix} +D \\ +4E \end{matrix}\right\} z.\overline{z-1} \left.\begin{matrix} +E \\ +5F \end{matrix}\right\} z.\overline{z-1}.\overline{z-2} \; z.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Quæ est forma desiderata.

Hem vero harum Operationum fundamentum. Quantitas reducenda, per multiplicationem reducatur ad Potestates Indeterminatæ z: dein operare ad modum Exempli sequentis. Sit $\overline{z-3} \times z \times \overline{z+1} \times \overline{z+4}$ quantitas reducenda; finge

$$\overline{z-3} \times z \times \overline{z+1} \times \overline{z+4} = az.\overline{z-1}.\overline{z-2}.\overline{z-3} + bz.\overline{z-1}.\overline{z-2} + cz.\overline{z-1} + dz.$$

Ubi maximus numerus Factorum in quantitate resoluta æqualis est numero eorundem in quantitate resolvenda. Reducatur utraque quantitas ad Potestates Indeterminatæ, factâ multiplicatione, & habebitur

$$z^4 + 2z^3 - 11z^2 - 12z = az^4 \left.\begin{matrix} -6a \\ +b \end{matrix}\right\} z^3 \left.\begin{matrix} +11a \\ -3b \\ +c \end{matrix}\right\} z^2 \left.\begin{matrix} -6a \\ +2b \\ -c \\ +d \end{matrix}\right\} z.$$

Atque comparando Terminos homologos, obtinebimus $a = 1$, $b - 6a = 2$, $c - 3b + 11a = -11$, $d - c + 2b - 6a = -12$; ex quibus eruitur $a = 1$, $b = 8$ $c = 2$, $d = -20$; hinc fit quantitas proposita

$$z^4 + 2z^3 - 11z^2 - 12z = z.\overline{z-1}.\overline{z-2}.\overline{z-3} + 8z.\overline{z-1}.\overline{z-2} + 2z.\overline{z-1} - 20z.$$

Et ad eundem prorsus modum procedere licet in aliis casibus. Brevitatis autem gratiâ, accipe regulam sequentem. Divide unitatem per Terminos hujus progressionis continuè, $n-1$, $n-2$, $n-3$, $n-4$, &c. hoc est, divide unitatem per $n-1$, & Quotum prodeuntem per $n-2$, & novissimum Quotum per $n-3$, & sic porro. Tum Quotos omnes sic prodeuntes dispone regulariter in Tabula ut vides, rejectis Dignitatibus ipsius n, & retentis solummodo Coefficientibus, utpote qui soli huic negotio sunt utiles; atque habebis

Tabulam priorem.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

Affume jam pro Coefficientibus, numeros in columnis defcendentibus, & habebis valores Dignitatum fequentes.

$$z = z,$$
$$z^2 = z + z.\overline{z-1},$$
$$z^3 = z + 3z.\overline{z-1} + z.\overline{z-1}.\overline{z-2},$$
$$z^4 = z + 7z.\overline{z-1} + z.\overline{z-1}.\overline{z-2} + z.\overline{z-1}.\overline{z-2}.\overline{z-3},$$
$$z^5 = z + 15.z.\overline{z-1} + 25z.\overline{z-1}.\overline{z-2} + 10z.\overline{z-1}.\overline{z-2}.\overline{z-3} + z.\overline{z-1}.\overline{z-2}.\overline{z-3}.\overline{z+4}, \&c.$$

Hac itaque Tabulâ femel habitâ, quantitas quælibet reducitur ad formam quæfitam abfque tædio computi. Proponatur ea hactenus reducta $z^4 + 2z^3 - 11z^2 - 12z$. Excerpe valores Dignitatum ex Tabula, eofque ducito refpective in fuos Coefficientes -12, -11, $+2$, & 1, atque obtinebis

$$\begin{aligned}
-12z &= -12z,\\
-11z^2 &= -11z - 11z.\overline{z-1},\\
+2z^3 &= +2z + 6z.\overline{z-1} + 2z.\overline{z-1}.\overline{z-2},\\
+z^4 &= +z + 7z.\overline{z-1} + 6z.\overline{z-1}.\overline{z-2} + z.\overline{z-1}.\overline{z-2}.\overline{z-3},\\
\hline
z^4 + 2z^3 - 11z^2 - 12z &= -20z + 2z.\overline{z-1} + 8z.\overline{z-1}.\overline{z-2} + z.\overline{z-1}.\overline{z-2}.\overline{z-3}.
\end{aligned}$$

Et valores membrorum in unam fummam collecti dant valorem totius ut jam prodiit. Notandum Seriem infinitam conflatam ex Dignitatibus

tatibus Indeterminatæ aſcendentibus reduci non poſſe generaliter in aliam prædictæ formæ: nam quiſque Coefficiens eſſet Series infinita. In Seriebus autem finitis, res ſuccedit ut ſupra oſtenſum eſt.

Series etiam alterius formæ ſimiliter reducuntur. Fingamus enim eſſe quantitatem quamvis quæſitam

$$T = \frac{A}{z} + \frac{B}{z.\overline{z+1}} + \frac{C}{z.\overline{z+1}.\overline{z+2}} + \frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Dein ſi occaſio poſtulet inquirere valorem ipſius T ſucceſſivum, ſcribe $z+1$ pro z, & emerget valor ſucceſſivus

$$T' = \frac{A}{z+1} + \frac{B}{\overline{z+1}.\overline{z+2}} + \frac{C}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Et ut hæc Series reducatur ad formam prioris, operor ad modum ſequentem,

$$\frac{A}{z+1} \ldots\ldots = \frac{A}{z} - \frac{A}{z.\overline{z+1}},$$

$$\frac{B}{\overline{z+1}.\overline{z+2}} \ldots = * . \frac{B}{z.\overline{z+1}} - \frac{2B}{z.\overline{z+1}.\overline{z+2}},$$

$$\frac{C}{\overline{z+1}.\overline{z+2}.\overline{z+3}} \ldots = * \ldots * \ldots \frac{C}{z.\overline{z+1}.\overline{z+2}} - \frac{3C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}},$$

$$\frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} = * \ldots * \ldots\ldots * \ldots \frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Et habeo $T' = \frac{A}{z} + \frac{B-A}{z.\overline{z+1}} + \frac{C-2B}{z.\overline{z+1}.\overline{z+2}} + \frac{D-3C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$

Ubi nunc Denominatores iidem ſunt atque in valore ipſius T; eaque de cauſa, inſtituere licet comparationem Terminorum prout occaſio poſtulat. Hujuſmodi autem Operationes ſic demonſtrantur. Pone $\frac{1}{z+1} = \frac{1}{z} - \frac{a}{z.\overline{z+1}}$, exiſtente a quantitate ſtatim inveniendâ; tum ducendo in Denominatorem $z.\overline{z+1}$, proveniet $z = z+1-a$, ſive delendo utrobique z, $0 = 1-a$, & $a = 1$; quare ſubſtituendo unitatem pro a, habebitur $\frac{1}{z+1} = \frac{1}{z} - \frac{1}{z.\overline{z+1}}$. Similiter fingo $\frac{1}{\overline{z+1}.\overline{z+2}} = \frac{1}{z.\overline{z+1}} - \frac{a}{z.\overline{z+1}.\overline{z+2}}$, & ducendo in Denominatorem, erit $z = z+$

$2-a$, ſive $a=2$, atque exinde $\frac{1}{z+1.z+2}=\frac{1}{z.z+1}-\frac{2}{z.z+1.z+2}$.

Proponatur jam

$$\frac{A}{z+2}+\frac{B}{z+2.z+3}+\frac{C}{z+2.z+3.z+4}+\frac{D}{z+2.z+3.z+4.z+5}+\&c.$$

quam reducere oportet in aliam debitæ formæ. Inſtituatur operatio ut mox oſtenſum, & invenies

$$\frac{A}{z+2}\text{ - - - - - - - }=\frac{A}{z}-\frac{2A}{z.z+1}+\frac{2A}{z.z+1.z+2},$$

$$\frac{B}{z+2.z+3}\text{ - - - - - }=*-\frac{B}{z.z+1}-\frac{4B}{z.z+1.z+2}+\frac{6A}{z.z+1.z+2.z+3};$$

$$\frac{C}{z+2.z+3.z+4}\text{ - - }=*\text{ - - - }*\text{ - - }\frac{C}{z.z+2.z+3}-\frac{6C}{z.z+1,z+2.z+3}+\&c.$$

$$\frac{D}{z+2\ z+3.z+4.z+5}=*\text{ - - - }*\text{ - - - - }*\text{ - - - - }\frac{D}{z\ z+1.z+2.z+3}-\&c.$$

Atque Series propoſita, ſub debita forma invenietur

$$\frac{A}{z}+\frac{B-2A}{z.z+1}+\frac{C-4B+2A}{z.z+1.z+2}+\frac{D-6C+6B}{z.z+1.z+2.z+3}+\&c.$$

Et hunc in modum inſtituetur reductio in aliis caſibus. Si fractio reducenda ſit $\frac{1}{z+1}$, erunt duo membra in ejuſdem valore, ut in Exemplo ſuperiore. Si ſit $\frac{1}{z+2}$, erunt tria, ut in Exemplo poſteriore. Et in genere in valore ipſius $\frac{1}{z+n}$ ad debitam formam reducti, numerus membrorum excedet numerum n unitate. Hìc tamen ſuppono n eſſe integrum & affirmativum; nam ſi ſit fractus vel negativus, valor fractionis $\frac{1}{z+n}$ excurret in infinitum.

Regula autem generalis pro hujuſmodi tranſmutationibus ea eſt quæ ſequitur. Duc Terminos hujus Progreſſionis n, $1+n$, $2+n$, $3+n$, &c. in ſe continuò, & Facta diſponantur in Tabula ſequente pro ratione Dignitatum numeri n, Coefficientibus tantùm reſervatis, & emerget

Tabula

Tabula posterior.

1								
1	1							
2	3	1						
6	11	6	1					
24	50	35	10	1				
120	274	225	85	15	1			
720	1764	1624	735	175	21	1		
5040	13068	13132	6769	1960	322	28	1	
40320	109584	105056	67284	22449	4536	546	36	1

&c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

Deinde sumendo Coefficientes ex columnis descendentibus, obtinebis valores Dignitatum,

$$\frac{1}{z^2} = \frac{1}{z.z+1} + \frac{1}{z.z+1.z+2} + \frac{2}{z.z+1.z+2z+3} + \frac{6}{z.z+1.z+2\,z+3.z+4} + \&c.$$

$$\frac{1}{z^3} = \frac{1}{z.z+1.z+2} + \frac{3}{z.z+1.z+2.z+3} + \frac{11}{z.z+1.z+2\,z+3.z+4} + \&c.$$

$$\frac{1}{z^4} = \frac{1}{z.z+1.z+2.z+3} \text{ in } 1 + \frac{6}{z+4} + \frac{35}{z+4.z+5} + \frac{225}{z+4\,z+5.z+6} + \&c.$$

Et ita porro in reliquis. Adeo ut habitâ Serie ex dignitatibus compositâ, ea semper reduci possit in aliam formæ desideratæ, ope hujus Tabulæ.

Vel propositâ Serie $\frac{A}{z^2} + \frac{B}{z^3} + \frac{C}{z^4} + \frac{E}{z^5} + \&c.$ sume Coefficientes ex columnis transversis, & ponantur

$$\begin{aligned} a &= A, \\ b &= A + B, \\ c &= 2A + 3B + C, \\ d &= 6A + 11B + 6C + D, \\ e &= 24A + 50B + 35C + 10D + E, \\ f &= 120A + 274B + 225C + 85D + 15E + F, \end{aligned}$$

&c.

Atque

Atque Series ex Dignitatibus composita transmutabitur in sequentem debitæ formæ

$$\frac{a}{z.\overline{z+1}}+\frac{b}{z.\overline{z+1}.\overline{z+2}}+\frac{c}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{d}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

Proponatur nunc Fractio $\frac{1}{zz+nz}$, primùm ope Divisionis resolvo eam in Seriem vulgarem $\frac{1}{z^2}-\frac{n}{z^3}+\frac{n^2}{z^4}-\frac{n^3}{z^5}+\frac{n^4}{z^6}-$ &c. unde est $A=1$, $B=-n$, $C=+n^2$, $D=-n^3$, $E=+n^4$, &c. & hisce valoribus substitutis, emergent $a=1$, $b=1-n$, $c=2-3n+n^2$, $d=6-11n+6n^2-n^3$, &c. atque adeo

$$\frac{1}{zz+nz}=\frac{1}{z.\overline{z+1}}+\frac{1-n}{z.\overline{z+1}.\overline{z+2}}+\frac{2-3n+n^2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{6-11n+6n^2-n^3}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

$$\text{Hoc est } \frac{1}{zz+nz}=\frac{1}{z.\overline{z+1}}+\frac{1-n}{z+2}A+\frac{2-n}{z+3}B+\frac{3-n}{z+4}C+\frac{4-n}{z+5}D+\&c.$$

Ubi quantitates A, B, C, D, &c. jam designant Terminos hujus Seriei more *Newtoniano*. Et patet Seriem abrumpere quotiescunque est n integer & affirmativus. In aliis Exemplis denotet etiam z minimum Factorem in Denominatore, & Series semper abrumpet hac methodo ubi ejus natura tulerit. Ut si Fractio sit $\frac{1}{x.\overline{x-3}.\overline{x+2}}$, pono $z=x-3$, minimo utique trium Factorum; tum erit $x=z+3$, & $x+2=z+5$: Atque evadet Fractio $\frac{1}{z.\overline{z+3}.\overline{z+5}}$, sive multiplicatione facta, $\frac{1}{z^3+8z^2+15z}$, quæ dividendo fit $\frac{1}{z^3}-\frac{8}{z^4}+\frac{49}{z^5}-\frac{272}{z^6}+\frac{1441}{z^7}-\frac{7448}{z^8}+\frac{37969}{z^9}-$ &c. Unde est $A=0$, $B=1$, $C=-8$, $D=+49$, $E=-272$, $F=+1441$, $G=-7448$, &c. Et exinde prodeunt $a=0$, $b=1$, $c=-5$, $d=12$, $e=-12$; at f & reliquæ sunt nihil: & per consequens abrumpit Series, existente accuratè

$$\frac{1}{z.\overline{z+3}.\overline{z+5}}=\frac{1}{z.\overline{z+1}.\overline{z+2}} \text{ in } 1-\frac{5}{z+3}+\frac{12}{\overline{z+3}.\overline{z+4}}-\frac{12}{\overline{z+3}.\overline{z+4}.\overline{z+5}}.$$

In fractione quavis $\frac{1}{z.\overline{z+a}.\overline{z+b}.\overline{z+c}.\&c.}$ sit z minimus Factorum, adeo, ut a, b, c, &c. sint affirmativi, & si sint quoque integri, Series terminabitur, aliàs excurret in infinitum. Ubi vero Series abrumpit, eadem inveniri potest plurimis modis idque elegantiùs quàm per regulam superiorem generalem: quippe a concinnitate alienissimum est,

primo

primo reducere Fractionem finitam in Seriem infinitam, ut poſtea ejus valor habeatur in Terminis numero finitis: quod nos hìc fecimus ut regula generalis illuſtraretur, non ut doceretur optima methodus quando Series abrumpit.

Si in Tabula priore excerpantur numeri ex columnis aſcendentibus, & ponantur

$$
\begin{aligned}
a &= \mathrm{A}, \\
b &= \mathrm{B} - \mathrm{A}, \\
c &= \mathrm{C} - 3\mathrm{B} + \mathrm{A}, \\
d &= \mathrm{D} - 6\mathrm{C} + 7\mathrm{B} - \mathrm{A}, \\
e &= \mathrm{E} - 10\mathrm{D} + 25\mathrm{C} - 15\mathrm{B} + \mathrm{A}, \\
f &= \mathrm{F} - 15\mathrm{E} + 65\mathrm{D} - 90\mathrm{C} + 31\mathrm{B} - \mathrm{A}, \\
&\&c.
\end{aligned}
$$

Tum Series hujus formæ $\frac{\mathrm{A}}{z.\overline{z+1}} + \frac{\mathrm{B}}{z.\overline{z+1}.\overline{z+2}} + \frac{\mathrm{C}}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$ migrabit in hanc $\frac{a}{z^2} + \frac{b}{z^3} + \frac{c}{z^4} + \frac{d}{z^5} + \&c.$ quæ ex Dignitatibus componitur.

In hiſce Tranſmutationibus nullam habuimus rationem Termini $\frac{\mathrm{A}}{z}$, quoniam is abſque ulla Tranſmutatione ambiguè pertinet tam ad Seriem Dignitatum quam ad eam Factorum.

PARS PRIMA.

DE

Summatione Serierum.

IN hac parte prima conatus ſum abbreviare computos in Quadratura Curvarum, & in Problematibus etiam difficilioribus, idque perveniendo ad valores Serierum infinitarum magis expeditè quàm per ſimplicem additionem Terminorum ut vulgò fit. Hæc quidem in citò convergentibus abunde rem conficit, neque ulteriore opus eſt artificio: attamen ubi lentè convergunt immenſus plerumque labor requiritur, & eo quidem major quo minor eſt convergentia; quod ſi lentiſſimè approximent, evadunt penitus intractabiles. Etenim notiſſimum eſt nonnunquam opus eſſe ultra mille numero Terminos, ut ſumma habeatur juſta ad duas treſve figuras. Monſtrabimus ergo in ſequentibus methodum expeditam tranſmutandi eas omnium lentiſſime convergentes in alias celerrime approximantes; ex quibus ſcilicet ſummæ ſupputari queunt minimo labore ad plurima figurarum loca.

Abrumpent quidem Series tranſmutatæ ubi ſummandæ ſunt ſummabiles; eoque in caſu tranſmutatio evadet ſummatio. Sed de Seriebus ſummabilibus minus ſollicitus ſum eaſque obiter tantum attingo, utpote quæ in Quadraturis raro prodire ſolent. Hic enim non operam collocavi in eruendis Seriebus inutilibus quæ per obvia Theoremata ſunt

ſunt ſummabiles, ſed in eruendis Theorematibus per quæ Series utiles expedite poſſint ſummari ad tot figurarum loca quot uſus quicunque requirunt.

De Seriebus ſimplicioribus.

Non tantum Seriei convergentia ſed & ſimplicitas plurimum confert ad contrahendos calculos. Quapropter priuſquam ad Tranſmutationes accedamus, ſciendum eſt Series *Newtonianas* in Tractatu de Quadratura Curvarum, non ſolùm abrumpere ubi rei natura tulerit, ſed & eſſe omnium ſimpliciſſimas, ubi excurrunt in infinitum, atque eapropter præferendas eſſe iis quæ inveniuntur methodo vulgari, ſcilicet reducendo Ordinatas in Series convergentes, ut exinde computentur Areæ.

Sit $x^{\theta-1} \times \overline{e+fx^{\eta}}|^{\lambda-1}$ Ordinata Curvæ, in qua x eſt Abſciſſa, e & f Coefficientes, atque $\theta-1$, $\lambda-1$, & η indices Dignitatum: pone $r = \frac{\theta+\eta}{\eta}$, $s = \frac{\theta+\lambda\eta}{\eta}$, & juxta *Newtonum* Area erit

$$\frac{x^{\theta}}{\theta e} \times \overline{e+fx}|^{\lambda} - \frac{s}{r} A \frac{fx^{\eta}}{e} - \frac{s+1}{r+1} B \frac{fx^{\eta}}{e} - \frac{s+2}{r+2} C \frac{fx^{\eta}}{e} - \frac{s+3}{r+3} D \frac{fx^{\eta}}{e} - \&c.$$

Ubi A, B, C, D, &c. deſignant Terminos, quemque in ſuo ordine ab initio: ſcilicet eſt $A = \frac{x^{\theta}}{\theta e} \times \overline{e+fx^{\eta}}|^{\lambda}$, $B = -\frac{s}{r} A \frac{fx^{\eta}}{e}$, $C = -\frac{s+1}{r+1} B \frac{fx^{\eta}}{e}$, & ſic deinceps. Proponatur jam inventio Arcûs ex dato Sinu recto x, vel quod perinde eſt, Quadratura Curvæ cujus Ordinata eſt $\frac{1}{\sqrt{1-xx}}$: hæc ad debitam formam reducta fit $x^{0} \times \overline{1-x^{2}}|^{-\frac{1}{2}}$, quæ comparata cum Ordinata generali dat $e=1$, $f=-1$, $\eta=2$, $\theta-1=0$, $\lambda-1=-\frac{1}{2}$; adeoque $\theta=1$, $\lambda=\frac{1}{2}$: et exinde $r=\frac{3}{2}$, $s=1$; quibus valoribus in Theoremate ſubſtitutis, oritur pro Arcu Series

$$x\sqrt{1-xx} + \tfrac{2}{3} A x^{2} + \tfrac{4}{5} B x^{2} + \tfrac{6}{7} C x^{2} + \tfrac{8}{9} D x^{2} + \tfrac{10}{11} E x^{2} + \&c.$$

At ſi Ordinata propoſita prius reſolvatur in Seriem per Theorema *Newtoni* pro evolvendo Binomio, dein capiatur Fluens cujuſque Termini, prodibit pro eodem Arcu Series

$$x + \frac{1\times1}{2\times3} Ax^{2} + \frac{3\times3}{4\times5} Bx^{2} + \frac{5\times5}{6\times7} Cx^{2} + \frac{7\times7}{8\times9} Dx^{2} + \frac{9\times9}{10\times11} Ex^{2} + \&c.$$

Unde patet priorem longe ſimpliciorem eſſe & proinde facilius continuari in infinitum. Exempli gratiâ, ſi Arcus quæſitus ſit octava pars totius Circum-

Circumferentiæ, ejus Sinus x erit æqualis $\sqrt{\frac{1}{2}}$, quo ſubſtituto, evadet Series

Prior $\frac{1}{2} + \frac{1}{3}A + \frac{2}{5}B + \frac{3}{7}C + \frac{4}{9}D + \frac{5}{11}E +$ &c.

Poſterior $\sqrt{\frac{1}{2}} + \frac{1}{12}A + \frac{9}{40}B + \frac{25}{84}C + \frac{49}{144}D + \frac{81}{220}E +$ &c.

Quo in caſu prior duplici de cauſa eſt præferenda, tum quoniam per Factores ſimpliciores producitur, tum quia libera eſt a numero ſurdo qui reperitur in poſteriore. Attamen ubi x eſt quantitas rationalis, & ſimul $x\sqrt{1-xx}$ irrationalis, Series poſterior eligenda eſt, modo ſit x magnitudinis uſque adeo contemnendæ quæ efficiat Seriem celerrime convergentem; hac enim ratione evitatur extractio Radicis quadratæ. Inſuper ſi ſit $x = 1$, neceſſe habemus recurrere ad ſecundam, quoniam in illo caſu evaneſcit quantitas $\sqrt{1-xx}$, in quam prima multiplicatur.

De Seriebus quæ celerius convergunt.

Ubi quantitas indeterminata, creſcente eâ quæ deſideratur, cito evadit prægrandis, & tandem infinite magna; Termini Seriei ex illa compoſitæ erunt alternatim negativi & affirmativi, & approximabunt lentius quam ubi indeterminata ultra datam magnitudinem creſcere nequit. Ut ſi quæratur Area vel Arcus circularis, melius eſt adhibere Sinum rectum qui nequit eſſe major Radio, quam Tangentem quæ cito excreſcit in immenſam longitudinem; ut olim obſervavit *Newtonus.* Et e contra præferendæ ſunt Tangentes in Hyperbola, utpote quæ datam magnitudinem ſuperare nequeunt, ſed continentur intra determinatos limites, eoſque ſatis arctos. Sed quæ hic diximus, non impediunt quo minus Area vel Arcus, magnitudinis mediocris vel contemnendæ, poſſit exquiri prout cuique viſum fuerit: nam differentia tantùm eſt notabilis in iis caſibus in quibus quantitates quæſitæ ſunt magnæ. Atque Series quarum Termini ſunt per vices negativi & affirmativi ſunt magis tractabiles quam alteræ, ubi de Summatione agitur. Quæ autem hic dicta ſunt de Curvis binomialibus, obtinent quoque in iis ſuperiorum nominum.

Verum quidem eſt Series celeriter convergentes multifariam inveniri poſſe, adhibita Methódo differentiali *Newtoniana.* Sed quo magis convergunt eo magis ſolent eſſe compoſitæ: quapropter præfero ſimpliciores etiamſi lentius convergentes.

De Summis successivis.

Per Summam successivam intelligo quantitatem quæ succedit Summæ omnium Terminorum, quando Termini subsequentes deveniunt in loca antecedentium. Ut si Summa sit $T+T'+T''+T'''+T^{iv}+T^{v}+$ &c. scribe Terminos posteriores pro prioribus, & habebis Summam successivam $T'+T''+T'''+T^{iv}\ T^{v}$ &c. in qua denuo si substituantur Termini consequentes pro antecedentibus, proveniet Summa $T''+T'''+T^{iv}+T^{v}+T^{vi}+$ &c. quæ succedit novissimæ; & sic deinceps.

Hinc si S, S', S'', S''', &c. denotent Summas successivas, erunt

$$\begin{aligned} S &= T+T'+T''+T'''+T^{iv}+\text{\&c.} \\ S' &= \text{- -}\ T'+T''+T'''+T^{iv}+\text{\&c.} \\ S'' &= \text{- - - - -}\ T''+T'''+T^{iv}+\text{\&c.} \\ S''' &= \text{- - - - - - - - -}\ T'''+T^{iv}+\text{\&c.} \end{aligned}$$

Hoc est, si a Serie quavis infinita subducatur primus Terminus, & a residua subducatur etiam primus Terminus, dein ab ea quæ relinquitur rursus subducatur primus Terminus, & sic ad libitum; Series hoc modo gradatim mulctatæ primis Terminis, erunt Summæ successivæ; hoc est,

$$S'=S-T,\ S''=S'-T',\ S'''=S''-T'',\ \text{\&c.}$$

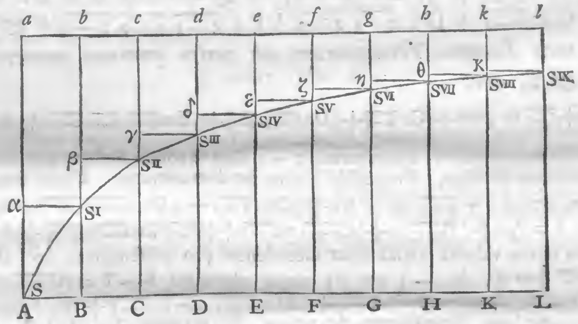
$$\begin{aligned} \text{Vel } S' &= S-T, \\ S'' &= S-T-T', \\ S''' &= S-T-T'-T'', \\ S^{iv} &= S-T-T'-T''=T''', \\ &\text{\&c.} \end{aligned}$$

Hic locuti sumus de Summis Terminorum omnium in infinitum, quæ incipiunt ad datum quemvis Terminum; nam quicunque Terminus sit T, S erit Summa ejus & omnium sequentium, item S' erit Summa ipsius T' & omnium reliquorum. Hæc quidem obtinent ubi agitur de Summa Terminorum numero infinitorum: Attamen ubi agitur de Summa Terminorum quorum numerus est finitus, S erit Summa Terminorum omnium abinitio ad datum quemvis Terminum T, & S' erit Summa eorundem Terminorum dempto T, atque S'' erit Summa eorundem demptis duobus T, T', & sic in reliquis.

Hinc in Summatione Terminorum a dato quovis Termino ad infinitum, si z sit longitudo Abscissæ quæ correspondit Summæ S; tum $z+1$, $z+2$, $z+3$, &c. erunt longitudines ejusdem respective correspondentes Summis successivis S', S'', S''', &c. Et e contra in Summatione

matione Terminorum a dato quovis usque ad initium Seriei, longitudines $z-1$, $z-2$, $z-3$, &c. respondebunt Summis S', S'', S''', &c. dummodo ipsi S respondeat Abscissa z. Nam in primo casu distantiæ Summarum ab initio perpetuo increscunt incremento Abscissæ, decrescuntque eodem decremento in posteriore.

Sit $S\,S'$ Curva quævis cujus Asymptotos est $a\,b$, eique parallela Abscissa AB. Dividatur Abscissa in partes innumeras inter se æquales AB, BC, CD, &c. Et a punctis divisionum A, B, C, D, &c. erigantur perpendicula ad Asymptoton, decussantia Curvam in punctis S, S', S'', &c. Asymptoton vero in a, b, c, &c. A punctis S', S'', S''', &c. ad Ordi-



natas proxime antecedentes ducantur $S'\alpha$, $S''\beta$, $S'''\gamma$, $S^{iv}\delta$, &c. parallelæ Abscissæ; adeo ut $S\alpha$, $S'\beta$, $S''\gamma$, $S'''\delta$, &c. sint differentiæ Ordinatarum tam earum quæ extenduntur a Curva ad Asymptoton, quam illarum quæ a Curva ad Abscissam porriguntur. Igitur Ordinatæ interceptæ inter Curvam & Asymptoton exponent Summas, & differentiæ pergendo ab iisdem in infinitum exponent Terminos. Hoc est, si $e\,S^{iv}$ designet Summam, succedentes erunt $f\,S^{v}$, $g\,S^{vi}$, $h\,S^{vii}$, &c. quarum differentiæ $\epsilon\,S^{iv}$, $\zeta\,S^{v}$, $\eta\,S^{vi}$, &c. in infinitum sunt Termini quorum Summa est $e\,S^{iv}$. Et similiter si $E\,S^{iv}$, $D\,S'''$, $C\,S''$, &c. designent Summas successivas, quarum prima est $E\,S^{iv}$, differentiæ antecedentes $\delta\,S'''$, $\gamma\,S''$, &c. exponent Terminos numero finitos pergentes ab Ordinata $E\,S^{iv}$ usque ad initium Seriei. Summatio igitur Serierum redu-

reducitur ad inventionem Ordinatarum ex datis earum Differentiis. Sed notandum oportere Summam ultimam esse nihil in utroque casu; quod semper fiet quando Curva transit per punctum A in Abscissa, & simul habet *ab* pro Asymptoto. Hæc cautio adhibenda est ut Summæ investigandæ per methodos tradendas, sint veræ, correctione minime indigentes, ut sæpissime fit in quadratura Curvarum.

Propositio I.

Si Termini alicujus Seriei formentur scribendo numeros 1, 2, 3, 4, 5, *&c. pro* z, *in Quantitate* $A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} +$ *&c. tum Summa Terminorum ab initio quorum numerus est* z, *erit*

$$Az + \overline{z+1} \text{ in } \tfrac{1}{2}Bz + \tfrac{1}{3}Cz.\overline{z-1} + \tfrac{1}{4}Dz.\overline{z-1}.\overline{z-2} + \tfrac{1}{5}Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Ubi notandum est quantitatem $z+1$ duci in totam Seriem quæ eam immediate sequitur. Propositio autem sic demonstratur. Finge Summam $S = Az + \overline{z+1}$ in $\frac{1}{2}Bz + \frac{1}{3}Cz.\overline{z-1} + \frac{1}{4}Dz.\overline{z-1}.\overline{z-2} + \&c.$ sive $S = Az + \frac{1}{2}B\overline{z+1}.z + \frac{1}{3}C\overline{z+1}.z.\overline{z-1} + \frac{1}{4}D\overline{z+1}.z.\overline{z-1}.\overline{z-2} + \&c.$ Dein scribe valores variabilium succedentes pro presentibus; hoc est, $S-T$ pro S; & $z-1$ pro z; atque obtinebis $S-T = A\overline{z-1} + \frac{1}{2}Bz.\overline{z-1}. + \frac{1}{3}Cz.\overline{z-1}.\overline{z-2} + \frac{1}{4}Dz.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$ subducito nunc hanc Æquationem de priore, & manebit $T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + \&c.$ Unde e contra si detur hic valor Termini ut in Propositione, Summa erit ea quæ assignata est. Insuper hæc Summa evadit nihil quando est z nihil: adeoque constat Theorema. Q. E. D.

Exemplum I.

Detur Series numerorum naturalium 1, 2, 3, 4, 5, 6, &c. hi formantur scribendo 1, 2, 3, &c. pro z, in ipsa quantitate z, quam itaque comparans cum Termino in Theoremate, erit $A=0$, $B=1$, & C, D, E, & sequentes nihil; quibus valoribus substitutis prodit Summa $\overline{z+1}$ in $\frac{1}{2}z$, sive $\dfrac{zz+z}{2}$ pro aggregato tot Terminorum Seriei

propo-

propositæ quot sunt unitates in z. Ut si sit $z=6$, proveniet $\frac{36+6}{2}=21$ pro Summa primorum sex Terminorum.

Exemplum II.

Detur jam Series numerorum imparium 1, 3, 5, 7, 9, 11, &c. hi formantur scribendo 1, 2, 3, 4, &c. in Quantitate $2z-1$, id est, $-1+2z$, quæ comparata cum valore Termini generali, dat $A=-1$, $B=2$, C, D, E, &c. nihil; quibus scriptis in Summa, emergit $-z+\overline{z+1}$ in $\frac{2z}{2}$, vel zz pro aggregato tot Terminorum quot numerat z. Atque ita quidem se res habet in casu præsente, nam Summæ successivæ sunt Quadrata numerorum naturalium.

Exemplum III.

Summanda sit Series Quadratorum 1, 4, 9, 16, 25, 36, 49, &c. quæ formantur expressione zz. Per ea quæ in Introductione explicantur, quantitas zz reducta ad formam Theorematis, evadit $z+z.\overline{z-1}$; adeoque $A=0$, $B=1$, $C=1$, & inde fit Summa $\overline{z+1}$ in $\frac{z}{2}+\frac{z.\overline{z-1}}{3}$, hoc est $\frac{z.\overline{z+1}.\overline{2z+1}}{6}$. Exempli causâ, scribe 7 pro z, & habebis $\frac{7.8.15}{6}=140$, quod est aggregatum septem Terminorum.

Exemplum IV.

Proponantur nunc Quadrata numerorum imparium 1, 9, 25, 49, 81, 121, 169, &c. eadem formantur scribendo 1, 2, 3, 4, &c. successive in expressione $1+4zz-4z$, quæ sic scripta $1+4z.\overline{z-1}$, dat $A=1$, $B=0$, $C=4$, D, E, &c. nihil: & hisce substitutis, prodit Summa $z+\overline{z+1}$ in $\frac{4}{3}z.\overline{z-1}$, sive $\frac{4z^3-z}{3}$.

Exemplum V.

Si dentur Cubi 1, 8, 27, 64, 125, 216, &c. quos assignat z^3, reducatur z^3 ad debitam formam $z+3z.\overline{z-1}+z.\overline{z-1}.\overline{z-2}$; eritque $A=0$, $B=1$, $C=3$, $D=1$, reliquæque erunt nihil; & propterea est

eſt Summa $\overline{z+1}$ in $\frac{1}{4}z + z.\overline{z-1} + \frac{1}{4}z.\overline{z-1}.\overline{z-2}$, quæ concinnata evadit $\frac{z^2}{4} \times \overline{z+1}^2$. Et hinc conſtat Summas horum Cuborum eſſe Quadrata numerorum 1, 3, 6, 10, 15, &c. ſcilicet Triangularium.

SCHOLION.

Hujuſmodi Series facilius ſummantur per Terminorum Differentias: deſignent enim A, A2, A3, &c. Seriem ſummandam; collige Terminorum differentias primas B, B2, B3, &c. ſecundas C, C2, C3, &c. tertias D, D2, &c. & ſic porro uſque dum perventum fuerit ad ultimam quæ hic eſt E: & Summa Terminorum cujus numerus eſt z, erit

A A2 A3 A4 A5
B B2 B3 B4
C C2 C3
D D2
E

$$A\frac{z}{1} + B\frac{z}{1} \times \frac{z-1}{2} + C\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + D\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} + \&c.$$

Sed notandum eſt quod differentiæ ſumi debeant auferendo priores de poſterioribus, hoc eſt, ponendo B = A2 — A, B2 = A3 — A2, &c. tum C = B2 — B, &c. Hujus autem demonſtratio pendet ex methodo differentiali *Newtonianâ.*

Proponantur Series 1, —1, 0, 8, 27, 61, 114, 119, &c. & collectis differentiis ad modum ſupra expoſitum, invenietur A = 1, B = —2, C = 3, D = 4, reliquæ autem nihil ſunt, adeoque evadit ſumma

1, —1, 0, 8, 27, 61, 114, 190,
—2, 1, 8, 19, 34, 53, 76,
3, 7, 11, 15, 19, 23,
4, 4, 4, 4, 4,

$$\frac{z}{1} - 2\times\frac{z}{1}\times\frac{z-1}{2} + 3\times\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3} + 4\times\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3}\times\frac{z-3}{4},$$

quæ in ordinem redacta fit $\frac{\overline{z-3}.\overline{z-2}.z.\overline{z+2}}{6}$. Series autem formatur ſcribendo 0, 1, 2, 3, 4, &c. in quantitate $\frac{4z^3 - 3zz - 13z + 6}{6}$.

PROPO-

PROPOSITIO II.

Si Termini cujuscunque Seriei formentur scribendo numeros quosvis unitate differentes in Quantitate

$$\frac{A}{z.\overline{z+1}} + \frac{B}{z.\overline{z+1}.\overline{z+2}} + \frac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} +$$

&c. Summa omnium Terminorum in infinitum, incipiens ad datum quemlibet Terminum erit

$$\frac{A}{z} + \frac{B}{2z.\overline{z+1}} + \frac{C}{3z.\overline{z+1}.\overline{z+2}} + \frac{D}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Pone Summam $S =$

$$\frac{A}{z} + \frac{B}{2z.\overline{z+1}} + \frac{C}{3z.\overline{z+1}.\overline{z+2}} + \frac{D}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Tum scribe valores ipsorum S & z, consequentes pro antecedentibus, hoc est $S - T$ pro S, & $z+1$ pro z, quoniam jam agitur de Terminis numero infinitis: & proveniet $S - T = \frac{A}{z+1} + \frac{B}{2.\overline{z+1}.\overline{z+2}} + \frac{C}{3.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{D}{4.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$ Hæc Æquatio subducta de superiore, relinquit $T = \frac{A}{z.\overline{z+1}} + \frac{B}{z.\overline{z+1}.\overline{z+2}} + \frac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$ Unde è contra si detur hic Terminus, Summa erit ea quæ in Propositione assignatur. Q. E. D.

Corollarium 1. Si Terminus sit $\frac{p}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}.\&c.}$, rejice ultimum Factorem, dein divide residuum per numerum Factorum qui relinquuntur, atque habebis Summam Terminorum. Sit Terminus $\frac{A}{z.\overline{z+1}}$; rejice ultimum Factorem $z+1$, & manebit $\frac{A}{z}$; quumque unicus sit Factor residuus z, erit $\frac{A}{z}$ Summa omnium Terminorum.

Sit

Sit jam Terminus conſtans ex tribus Factoribus $\frac{B}{z.\overline{z+1}.\overline{z+2}}$, rejice ultimum Factorem $z+2$, & reſtabit $\frac{B}{z.\overline{z+1}}$, quæ diviſa per 2, numerum utique Factorum qui relinquuntur, exhibet $\frac{B}{2z.\overline{z+1}}$ pro Summa.

Similiter ſi ex Termino $\frac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$ ex quatuor Factoribus conflato, rejiciatur ultimus $z+3$, & reſiduum dividatur per 3, obtinebitur Summa $\frac{C}{3z.\overline{z+1}.\overline{z+2}}$.

Si Terminus ſit $\frac{A}{z}$, rejice Factorem z, & quoniam nullus reſtat, divide A per Nihil, & habebis pro Summâ Quantitatem infinite magnam, uti notum eſt. Et hac de re primus quod ſciam egit D. *Taylor* in Methodo Incrementorum. Eadem etiam fuſius & elegantiſſime traditur a D. *Nicol* in Actis Academiæ Regiæ Pariſienſis.

Corollarium 2. Per ea quæ hac ſuper materia habentur in Introductione, conſtat Terminum quempiam $\frac{A}{z.\overline{z+a}.\overline{z+b}.\overline{z+c}.\,\&c.}$ reſolvi ſemper poſſe in Terminos duos forte aut plures ſummabiles & numero finitos, quando a, b, c, &c. ſunt numeri integri; in illo igitur caſu Series ſummabitur. Ut ſi Terminus ſit $\frac{1}{z.\overline{z+3}}$, idem reſolvitur in tres Terminos ſummabiles $\frac{1}{z.\overline{z+1}} - \frac{2}{z.\overline{z+1}.\overline{z+2}} + \frac{2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$. Unde per Corollarium præcedens, erit Summa $\frac{1}{z} - \frac{1}{z.\overline{z+1}} + \frac{2}{3z.\overline{z+1}.\overline{z+2}}$, quæ ſimul juncta ſunt $\frac{3z^2+6z+2}{3z^3+9z^2+6z}$. Et ſimiliter ſi Terminus ſit hujus formæ $\frac{a+bz^2+cz^3+dz^4+\&c.}{z.\overline{z+a}.\overline{z+b}.\overline{z+c}.\overline{z+d}.\,\&c.}$ Series ſummabitur, dummodo a, b, c, d, &c. ſint integri, & numerus Factorum in Denominatore, ad minimum binario excedat altiſſimum ipſius z dimenſionem in Numeratore. Sed excipio caſus in quibus duo aut plures Factores in Denominatore ſibi invicem æquantur; in illis Series non ſunt ſummabiles.

EXEMPLUM

EXEMPLUM I.

Proponatur Series summanda $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\frac{1}{10.13.16}+\frac{1}{13.16.19}+$ &c. Termini hujus Seriei assignantur quantitate $\frac{1}{3z.\overline{3z+3}.\overline{3z+6}}$, uti patebit scribendo pro z successive $\frac{1}{3}$, $1\frac{1}{3}$, $2\frac{1}{3}$, $3\frac{1}{3}$, &c. id est, hâc $\frac{1}{27.z.\overline{z+1}.\overline{z+2}}$; unde fit Summa $\frac{1}{54z.\overline{z+1}}$. Nam in eadem scribendo pro z valorem suum primum $\frac{1}{3}$, prodibit $\frac{1}{24}$ pro Summa totius Seriei. Si scribatur pro z valor suus secundus $1\frac{1}{3}$, prodibit $\frac{1}{168}$ pro Summa totius Seriei dempto primo Termino. Si pro z scribatur valor suus tertius $2\frac{1}{3}$, prodibit $\frac{1}{420}$ pro Summa totius Seriei demptis primis duobus Terminis. Et sic in infinitum.

EXEMPLUM II.

Proponatur Series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\frac{1}{5.8}+$ &c. Termini hujus assignantur quantitate $\frac{1}{z.\overline{z+3}}$, in qua oportet scribi 1, 2, 3, 4, &c. successive pro z. Quantitas autem $\frac{1}{z.\overline{z+3}}$ reducitur in tres Terminos summabiles nempe $\frac{1}{z.\overline{z+1}}-\frac{2}{z.\overline{z+1}.\overline{z+2}}+\frac{2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$; Unde erit Summa $\frac{1}{z}-\frac{1}{z.\overline{z+1}}+\frac{2}{3z.\overline{z+1}.\overline{z+2}}$, sive $\frac{3z^2+6z+2}{3z.\overline{z+1}.\overline{z+2}}$. Jam si vis aggregatum omnium omnino Terminorum, substitue in Summa unitatem pro z, & obtinebis $\frac{3+6+2}{3.1.2.3}$, id est $\frac{11}{18}$ pro valore Seriei propositæ.

EXEMPLUM III.

Detur jam Series $\frac{1}{2.3.4.5}+\frac{4}{3.4.5.6}+\frac{9}{4.5.6.7}+\frac{16}{5.6.7.8}+\frac{25}{6.7.8.9}+$ &c. ubi Numeratores sunt Quadrata numerorum naturalium;

Terminus quilibet in genere assignabitur expressione $\frac{zz - 2z + 1}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$, existentibus 2, 3, 4, 5, &c. valoribus indeterminatæ successivis. Et quantitas illa resolvitur in tres Terminos summabiles scilicet $\frac{1}{z.\overline{z+1}}$ — $\frac{7}{z.\overline{z+1}.\overline{z+2}} + \frac{16}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$: & inde Summa $\frac{1}{z} - \frac{7}{2z.\overline{z+1}} + \frac{16}{3z.\overline{z+1}.\overline{z+2}}$, hoc est $\frac{6z^2 - 3z + 2}{6.z.\overline{z+1}.\overline{z+2}}$, in qua si substituas 2 pro z, habebis $\frac{5}{24}$ pro valore Seriei.

Exemplum IV.

Quæratur valor Seriei $\frac{1}{1.2.3.4.5} + \frac{27}{2.3.4.5.6} + \frac{125}{3.4.5.6.7} + \frac{343}{4.5.6.7.8} +$ &c. ubi Numeratores sunt Cubi numerorum imparium 1, 3, 5, 7, &c. Positis jam 1, 2, 3, 4, 5, &c. valoribus ipsius z successivis, Termini assignabuntur per expressionem $\frac{8z^3 - 12z^2 + 6z - 1}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}$, quæ resolvitur in $\frac{8}{z.\overline{z+1}} - \frac{84}{z.\overline{z+1}.\overline{z+2}} + \frac{386}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$ — $\frac{729}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}$. Igitur est Summa $\frac{8}{z} - \frac{84}{2z.\overline{z+1}} + \frac{386}{3z.\overline{z+1}.\overline{z+2}} - \frac{729}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$, hoc est, $\frac{96z^3 + 72z^2 + 80z - 3}{12z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$; in qua si scribas primum valorem ipsius z, id est, unitatem pro z, invenies $\frac{245}{288}$ esse valorem Seriei.

Exemplum V.

Proponatur Series $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} +$ &c. quam invenit Vicecomes *Brounker* pro Quadratura Hyperbolæ; Terminus quivis in genere assignatur per expressionem $\frac{1}{4z.\overline{z+\frac{1}{2}}}$, ubi valores ipsius z sunt $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c. Et quantitas reducta ad formam summabilem evadit

$$\frac{1}{4z.\overline{z+1}}+\frac{1}{8z.\overline{z+1}.\overline{z+2}}+\frac{1.3}{16z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{1.3.5}{32z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

Excurrit quippe in Seriem infinitam propterea quod differentia Factorum est numerus fractus in expressione assignante Terminos; hoc utique indicium est Seriem non esse summabilem. Regrediendo autem à Termino ad Summam, habebitur

$$\frac{1}{4z}+\frac{1}{16z.\overline{z+1}}+\frac{1.3}{48z.\overline{z+1}.\overline{z+2}}+\frac{1.3.5}{128z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\&c.$$

Quæ est Series eo citius convergens quo major est z. Sed ob faciliorem computum pone $A=\frac{1}{4z}$,

$$B=\frac{A}{2z+2},\ C=\frac{3B}{2z+4},\ D=\frac{5C}{2z+6},\ E=\frac{7D}{2z+8},\ F=\frac{9E}{2z+10},$$

&c. atque erit Summa, $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\frac{1}{5}E+$ &c. In qua si pro z substituatur primus ejus valor $\frac{1}{2}$, habebitur valor totius Seriei summandæ: si pro z substituatur ejus valor secundus, proveniet Summa omnium Terminorum dempto primo; si pro z substituatur ejus valor tertius, proveniet Summa omnium exceptis primis duobus, & sic porro. Substituo igitur pro z valorem suum decimum quartum $13\frac{1}{2}$, ut z sit satis magna ad efficiendum Seriem celeriter convergere; & habeo $A=\frac{1}{54}$, $B=\frac{1}{29}A$, $C=\frac{3}{31}B$, $D=\frac{5}{33}C$, $E=\frac{7}{35}D$, $F=\frac{9}{37}E$, &c. quo in casu Summa $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ &c. æquabitur $\frac{1}{27.28}+\frac{1}{29.30}+\frac{1}{31.32}+$ &c. toti utique Seriei summandæ demptis primis tredecim Terminis. Horum igitur Summam quæro per additionem, invenioque fore .674285961. Dein ut obtineam Summam reliquorum, calculo eruo A, B, C, D, &c. ad tot decimalium loca quot est animus; eosque inventos divido respective per 1, 2, 3, 4, 5, &c. ut vides.

A = .018518519	.018518519
B = - 638570	319285
C = - - 61797	20599
D = - - - 9363	2341
E = - - - 1873	375
F = - - - 455	76
G = - - - 128	18
H = - - - - 41	5
I = - - - - 14	1
	.018861219

Eoque

Eoque pacto obtineo .018861219 pro Summa Terminorum omnium post decimum tertium; hæc denique adjecta aggregato initialium prius invento, conficit .693147180 pro valore Seriei summandæ, id est, pro Logarithmo Hyperbolico binarii.

Quo plures Termini sub initio colliguntur, eo citius converget Series quæ dat Summam reliquorum, propter z tanto majorem. Atque hujus methodi præstantia in eo maxime enitet quod addendo Terminos aggregato initialium, z tot unitatibus augeatur, qua ratione Series transmutata ad libitum fere converget.

Quod autem in praxi impossibile sit assequi Summas harum Serierum per meram collectionem Terminorum, patebit ex computo sequente; ubi habetur Summa centum, mille, decies mille, & sic porro usque ad decies millies centena millia Terminorum.

Summa	100 - - - -	Terminorum	.690653446
	1000 - - - -		.692897242
	10000 - - -		.693122181
	100000 - - -		.693144680
	1000000 - -		.693146930
	10000000 - -		.693147155
	100000000 -		.693147178
	1000000000		.693147180

Ex hoc calculo constat centum Terminos dare Summam accuratam ad duas figuras; & gradatim decuplum Terminorum numerum colligendo, lucrari præterpropter unam tantummodo figuram: adeo ut si quis vellet eruere valorem hujus Seriei accuratum ad novem figurarum loca, nulla arte adhibita præter additionem, requirerentur circiter decies millies centena millia Terminorum. Et hæc Series convergit longe celerius quam aliæ quamplurimæ, quarum valores sunt Quantitates finitæ.

EXEMPLUM VI.

Summanda sit Series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} +$ &c. ubi Denominatores sunt Quadrata numerorum 1, 2, 3, 4, &c. & Terminus in genere est $\frac{1}{zz}$: hic reductus in formam summabilem, evadit

$$\frac{1}{z.\overline{z+1}} + \frac{1}{z.\overline{z+1}.\overline{z+2}} + \frac{1.2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{1.2.3}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Ergo Summa =

$$\frac{1}{z} + \frac{1}{2z.\overline{z+1}} + \frac{1.2}{3z.\overline{z+1}.\overline{z+2}} + \frac{1.2.3}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Quæ

Quæ ponendo $A=\frac{1}{z}$, $B=\frac{A}{z+1}$, $C=\frac{2B}{z+2}$, $D=\frac{3C}{z+3}$, $E=\frac{4D}{z+4}$, $+$ &c. evadit $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ &c. Jam si pro z substituatur ejus valor decimus tertius 13, habebitur Summa omnium Terminorum in Serie summanda post decimum secundum; quo in casu erit $A=\frac{1}{13}$, $B=\frac{1}{14}A$, $C=\frac{2}{15}B$, $D=\frac{3}{16}C$, $E=\frac{4}{17}D$, &c. Et $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ &c. erit Summa Terminorum $\frac{1}{169}+\frac{1}{196}+\frac{1}{225}+\frac{1}{256}+$ &c. Computus autem sic se habet

A = .076923077	.076923077
B = - 5494505	2747252
C = - - 732601	244200
D = - - 137363	34341
E = - - - 32321	6464
F = - - - 8978	1496
G = - - - 2835	405
H = - - - - 992	124
I = - - - - 378	42
K = - - - - 155	16
L = - - - - 67	6
M = - - - - 31	3
N = - - - - 15	1
	.079957427

Unde prodit .079957427 pro Summa Terminorum $\frac{1}{169}+\frac{1}{196}+\frac{1}{225}+$ &c. quæ adjecta aggregato duodecim initialium, sive 1.564976638, conficit 1.644934065 pro valore totius Seriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ &c. Convergit autem hæc Series minus celeriter ea *Brounkeri* in Exemplo priore.

PROPOSITIO III.

Si Termini cujusvis Seriei formentur scribendo numeros quoslibet unitate differentes pro z in Quantitate

$$x^{z+n} \text{ in } \frac{a}{z} + \frac{b}{z.\overline{z+1}} + \frac{c}{z.\overline{z+1}.\overline{z+2}} + \frac{d}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

erit Summa æqualis x^{z+n} *in*

$$\frac{a}{\overline{1-x}.z} + \frac{b-Ax}{\overline{1-x}.z.\overline{z+1}} + \frac{c-2Bx}{\overline{1-x}.z.\overline{z+1}.\overline{z+2}} + \frac{d-3Cx}{\overline{1-x}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} \&c.$$

Quantitates A, B, C, D, &c. designant Coefficientes Terminorum præcedentes eos in quibus reperiuntur; scilicet est $A = \frac{a}{1-x}$, $B = \frac{b-Ax}{1-x}$, $C = \frac{c-2Bx}{1-x}$, &c. & sic porro. Excipio autem casum in quo est x æqualis unitati: ubi hoc accidit, Series summabitur per Propositionem superiorem. Hem vero demonstrationem. Finge Summam

$$S = x^{z+n} \text{ in } \frac{A}{z} + \frac{B}{z.\overline{z+1}} + \frac{C}{z.\overline{z+1}.\overline{z+2}} + \frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

deinde scribe valores variabilium succedentes $S-T$ & $z+1$ pro antecedentibus S & z respective, atque habebis

$S-T = x^{z+n+1}$ in

$$\frac{A}{z+1} + \frac{B}{\overline{z+1}.\overline{z+2}} + \frac{C}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Hoc est, $S = x^{z+n}$ in

$$\frac{Ax}{z+1} + \frac{Bx}{\overline{z+1}.\overline{z+2}} + \frac{Cx}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{Dx}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

quæ reducta ad formam ipsius S, evadit

$S-T = x^{z+n}$ in

$$\frac{Ax}{z} + \frac{Bx-Ax}{z.\overline{z+1}} + \frac{Cx-2Bx}{z.\overline{z+1}.\overline{z+2}} + \frac{Dx-3Cx}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Subdu-

Subducito nunc valorem ipsius S — T a valore ipsius S, & relinquetur Terminus

$T = x^{z+n}$ in

$$\frac{A\overline{1-x}}{z} + \frac{B\overline{1-x}+Ax}{z.\overline{z+1}} + \frac{C\overline{1-x}+2Bx}{z.\overline{z+1}.\overline{z+2}} + \frac{D\overline{1-x}+3Cx}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Hic denique valor ipsius T collatus cum illo in Propositione, dat $A\overline{1-x} = a$, $B\overline{1-x} + Ax = b$, $C\overline{1-x} + 2Bx = c$, & ita porro. Quæ Æquationes exhibent valores Coefficientium ut supra. Quare valor Summæ recte assignatur. Q. E. D.

Exemplum I.

Summanda sit Series $1 + \frac{1}{3}t + \frac{1}{5}t^2 + \frac{1}{7}t^3 + \frac{1}{9}t^4 + \&c.$ Æquatio ad eandem est $T = t^{z-\frac{1}{2}}$ in $\frac{\frac{1}{2}}{z}$, nam scribendo $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c. successive pro z, provenient Termini Seriei. Comparando autem hunc Terminum cum illo in Theoremate; erit $x = t$, $n = -\frac{1}{2}$, $a = \frac{1}{2}$; at b, c, d, e, & Coefficientes reliquæ sunt nihil. Et hisce demum valoribus scriptis, oritur

$S = t^{z-\frac{1}{2}}$ in

$$\frac{\frac{1}{2}}{\overline{1-t}.z} + \frac{At}{\overline{t-1}.z.\overline{z+1}} + \frac{2Bt}{\overline{t-1}.z.\overline{z+1}.\overline{z+2}} + \frac{3Bt}{\overline{t-1}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} \&c.$$

Ponamus Exempli gratia esse $t = -1$; & Series summanda erit $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$ atque evadet Summa

$$S = \pm 1 \text{ in } \frac{1}{4z} + \frac{A}{2z.\overline{z+1}} + \frac{2B}{2z.\overline{z+1}.\overline{z+2}} + \frac{3C}{2z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

$$\text{Vel } S = \pm 1 \text{ in } \frac{1}{4z} + \frac{A}{2z+2} + \frac{2B}{2z+4} + \frac{3C}{2z+6} + \frac{4D}{2z+8} + \&c.$$

Ubi A, B, C, D, &c. jam designant totos Terminos more *Newtoniano*, & non amplius Coefficientes. Et unitas cum signo ambiguo in quam tota Series multiplicatur, erit affirmativa ubi $z - \frac{1}{2}$ est numerus par, & negativa ubi est impar. Collige jam duodecem Terminos initiales, vel quod perinde est, sex in hac Serie $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \&c.$

conjunctis

conjunctis binis quibuſque in priore: eorumque Summam reperies .7646006915. Dein ſcribe pro z ejus valorem decimum tertium $12\frac{1}{2}$, & obtinebis

$$S = \frac{1}{50} + \frac{1}{27}A + \frac{2}{29}B + \frac{3}{31}C + \frac{4}{33}D + \frac{5}{35}E + \&c.$$

quæ eſt Series ſimplex & celeriter convergens: decem etenim Termini dant $S = .0207974719$ ut ex computo appoſito liquet: & eâdem adjectâ aggregato initialium, provenit .7853981634 pro valore Seriei ſummandæ: ad quem tamen nunquam pervenire liceret per additionem Terminorum. Atque colligendo plures initiales, valor ipſius S longe celerius approximabit. Hujus itaque Propoſitionis ope, Circumferentia circuli produci poteſt minimo labore ad figuras quamplurimas, per Seriem hancce utcunque lente convergentem; id quod olim multum deſiderabat *Leibnitius*.

$$\begin{array}{r} .0200000000 \\ 7407407 \\ 510856 \\ 49438 \\ 5992 \\ 856 \\ 139 \\ 25 \\ 5 \\ 1 \\ \hline S = .0207974719 \end{array}$$

Periferia circuli obtinebitur quoque accuratiſſime per Seriem *Newtoni* ſequentem $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \&c.$ ubi Termini quique bini ſunt alternatim negativi & affirmativi. Idem etiam efficitur per hancce $1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \&c.$ in qua Denominatores conſtituunt progreſſionem numerorum naturalium, dempto quoque terno. Prior æquatur quartæ & poſterior tertiæ totius circumferentiæ parti, ex ſuppoſitione quòd horum Arcuum Chordæ ſint unitates. Priuſquam tamen tractentur per hanc Propoſitionem, diſpertiendæ ſunt utræque in duas; prior in haſce

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \frac{1}{21} + \&c.$$

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} - \frac{1}{23} + \&c.$$

Poſterior in haſce $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \&c.$

$$\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \frac{1}{17} + \&c.$$

Dein

Dein harum quatuor unaquæque feorfim confideranda eft, & operatio inftituenda ut in Exemplo fuperiore.

Exemplum II.

Si Series fit $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \frac{x^5}{9.10} +$ &c. Æquatio erit $T = x^z$ in $\frac{1}{4z.z - \frac{1}{2}}$, quæ refoluta in Seriem evadit

$T = x^z$ in

$$\frac{1}{4z.z+1} + \frac{3}{8z.z+1.z+2} + \frac{15}{16z.z+1.z+2.z+3} + \frac{105}{32z.z+1.z+2.z+3.z+4} + \&c.$$

Unde comparando membra, erit $n = 0$, $a = \frac{1}{4}$, $b = \frac{3}{8}$, $c = \frac{15}{16}$, $d = \frac{105}{32}$, $e = \frac{945}{64}$, &c.

Adeoque $S = x^z$ in

$$\frac{1}{4.1-x.z} + \frac{3-8Ax}{8.1-x.z.z+1} + \frac{15-32Bx}{16.1-x.z.z+1.z+2} + \frac{105-96Cx}{32.1-x.z.z+1.z+2.z+3} + \&c.$$

Cujus Seriei progreffio cuivis patet. Et ubi datur valor ipfius x in cafu quovis particulari, dabitur Summa pro libitu accurata; primo fcilicet addendo numerum fufficientem Terminorum initialium, eo fine ut z fit fatis magna ad efficiendum valorem ipfius S celeriter convergere. Et hifce præmiffis de Seriebus quarum Termini funt affignabiles, pergendum eft ad eas quæ determinantur per relationem Terminorum.

Propositio IV.

Data relatione inter Summas fucceffivas, invenire eam quæ eft inter Terminos.

In Æquatione definiente relationem inter Summas, fubftitue pro S′, S″, S‴, &c. fuos valores proprios S—T, S—T—T′, S—T—T′—T″, &c. & fic habebis Æquationem involventem Summam unicam S; in qua fcribe valores variabilium confequentes pro antecedentibus, & habebis Æquationem novam involventem Summam illam S: ope harum denique Æquationum eliminetur S, & ea quæ refultat monftrabit relationem Terminorum. Q. E. I.

Exemplum I.

Sit Æquatio ad Summas, $\overline{z-n}\,S = \overline{z-1}\,S'$; pro S' ſubſtitue ſuum valorem $S-T$, ac evadet Æquatio $\overline{n-1}\,S = \overline{z-1}\,T$; in qua ſcribe valores variabilium ſuccedentes pro antecedentibus, id eſt, $S-T$ pro S, T' pro T, & $z+1$ pro z; atque prodibit $\overline{n-1}\,S = \overline{n-1}\,T + zT'$; hanc de priore $\overline{n-1}\,S = \overline{z-1}\,T$ ſubducito, & manebit $\overline{z-n}\,T = zT'$: quæ eſt Æquatio ad Terminos Seriei.

Exemplum II.

Proponatur Æquatio ad Summas $S\times\overline{8zz+20z+9} + 3S'\times\overline{8zz+4z-3}$; ſubſtitue $S-T$ pro S', ac invenies $S = \frac{3}{32}\,T\times\frac{8zz+4z-3}{zz+z}$: dein juxta methodum differentialem ſcribe $S-T$ pro S, T' pro T, & $z+1$ pro z, ac proveniet $S-T = \frac{3}{32}\,T\times\frac{8zz+20z+9}{zz+3z+2}$; ope harum Æquationum extermina S, & habebitur $\overline{z+2}\,T + 3T'z = 0$, quæ eſt Æquatio exhibens relationem Terminorum.

Atque eadem methodo exterminantur tres, pluresve Summæ ſucceſſivæ.

Propositio V.

Invenire Series quotvis ſummabiles.

Æquatio ad Summas dabit Summam Terminorum, ea vero ad Terminos dabit Seriem; prior pro libitu aſſumitur, & ex ea deducitur poſterior per Propoſitionem ſuperiorem: habentur ergo Termini, eorumque Summa. Q. E. I.

Exemplum I.

Sit Æquatio ad Summas $\overline{z-n}\,S = \overline{z-1}\,S'$, ut in Exemplo primo Propoſitionis præcedentis; invenies eam ad Terminos eſſe $\overline{z-n}\,T = zT'$. Æquatio autem ad Summas, ſubſtituendo $S-T$ pro S', dabit Summam $S = \frac{z-1}{n-1}\,T$. Deſignent jam A, B, C, D, &c. Terminos hujus Seriei, & in Æquatione ad eoſdem ſcribe, m, $m+1$, $m+2$, $m+3$, &c. ſucceſ-

ſucceſſive pro z, exiſtente m numero quocunque integro vel fracto, negativo vel affirmativo; atque prodibunt relationes Terminorum, $B = \frac{m-n}{m} A$, $C = \frac{m-n+1}{m+1} B$, $D = \frac{m-n+2}{m+2} C$, $E = \frac{m-n+3}{m+3} D$, &c. Tum in Æquatione $S = \frac{z-1}{n-1} T$, ſcribe primum Terminum Seriei, id eſt, A pro T, & primum valorem ipſius z, id eſt, m pro z; atque invenies

$$S = \frac{m-1}{n-1} A = A + \frac{m-n}{m} A + \frac{m-n+1}{m+1} B + \frac{m-n+2}{m+2} C + \&c.$$

Ubi ſubſtituere licet quoſvis numeros pro m & n. Sit Exempli gratiâ $m=5$, $n=2$, $A=\frac{1}{12}$, & proveniet

$$\frac{1}{3} = \frac{1}{12} + \frac{3}{5} A + \frac{4}{6} B + \frac{5}{7} C + \frac{6}{8} D + \&c.$$

$$\text{id eſt, } \frac{1}{3} = \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \&c.$$

Ubi patet Terminos eſſe aſſignabiles; id quod ſemper eveniet quando n eſt numerus integer: tot vero erunt factores in Denominatoribus, quot ſunt unitates in n. Sic in præſente Exemplo, eſt $n=2$, & propter id ſunt duo factores in Denominatoribus Terminorum.

Sit jam $m=2$, $n=\frac{3}{2}$, $A=1$, & prodibit

$$S = 2 = 1 + \frac{1}{4} A + \frac{3}{6} B + \frac{5}{8} C + \frac{7}{10} D + \frac{9}{12} E + \&c.$$

$$\text{hoc eſt, } S = 2 = 1 + \frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + \frac{21}{512} + \&c.$$

Nunc autem conſtat Terminos non eſſe aſſignabiles, quoniam n eſt numerus fractus. Notandum Seriem abrumpere & eſſe numero Terminorum finitam quotieſcunque eſt $m-n$ nihil vel integer & negativus. Atque ſi $n-1$ ſit nihil vel numerus negativus valor Seriei erit infinite magnus: ut conſtat ex valore Summæ, nempe $\frac{m-1}{n-1} A$.

EXEMPLUM

EXEMPLUM II.

Sit Æquatio ad Summas $S\times\overline{8zz+20z+9}+3S'\times\overline{8zz+12z-3}$, ut in Exemplo noviſſimo Propoſitionis ſuperioris, ubi inventa eſt $S=\frac{3}{32}T\times\frac{8zz+4z-3}{zz+z}$, & relatio Terminorum erat $\overline{z+2}\,T+3zT'=0$. Et ſi uſurpetur unitas pro Termino primo, & primus valor ipſius z ponatur etiam unitas; obtinebitur

$$S=\frac{27}{64}=1-\frac{3}{3}+\frac{6}{9}-\frac{10}{27}+\frac{15}{81}-\frac{21}{243}+\&c.$$

Ubi Denominatores ſunt dignitates ternarii, Numeratores vero numeri triangulares. In hiſce Exemplis non expatiatus ſum in deducendis Seriebus ex Æquationibus quæ definiunt relationes Terminorum, quoniam hoc ex Introductione hactenus notum ſuppono.

SCHOLION.

Summatio Serierum in Methodo differentiali, reſpondet Quadraturæ Curvarum in Methodo Fluxionum; & eapropter in utraque oriri ſolent difficultates conſimiles, quæ hic ſunt enodandæ. Diximus Series hinc inde continuari poſſe in infinitum: exempli gratia Series $1+x+x^2+x^3+\&c.$ retrorſum continuata eſt $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\&c.$ Et hæ duæ ſimul junctæ conſtituunt unicam ex utraque parte excurrentem in infinitum, ſcilicet

$$\&c.+\frac{1}{x^4}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1+x+x^2+x^3+x^4+\&c.$$

Nam hi Termini ſunt in progreſſione Geometrica continua, antecedentes ad conſequentes ut unitas ad x. In Summatione hujus Seriei invenienimus $\frac{1}{1-x}=1+x+x^2+x^3+\&c.$ quod quidem verum eſt ubi x eſt minor unitate; attamen ſi ſit major unitate, hæc Series erit infinite magna, atque Summa $\frac{1}{1-x}$, non amplius erit Summa horum Terminorum; ſed mutato ſigno, æquabitur Seriei ad alteras partes excurrenti; hoc eſt, erit $\frac{-1}{1-x}$, vel $\frac{1}{x-1}=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\&c.$ Quod

ſi x ſit unitas, Summa erit $\frac{1}{1-1}$, & inde utraque pars Seriei erit infinite magna, utpote æqualis unitati infinities ſumptæ.

Ad eundem modum ſi Æquatio ad Seriem ſit $\overline{z-n}\,T = zT'$, & Series continuetur ex utraque parte in infinitum; una pars converget & altera diverget præterquam ubi eſt $n=1$; & Summa $\frac{z-1}{n-1}$ T ſemper æquabitur parti Seriei convergenti.

Haud ſecus in Quadraturis, ſi z^{-n} ſit Ordinata Curvæ Hyperbolicæ, fluens $\frac{1}{1-n}z^{1-n}$ exprimet partem Areæ ad has aut illas Ordinatæ partes jacentem prout n eſt minor aut major unitate: ubi tamen n eſt unitas, Area ex utraque parte Ordinatæ erit infinite magna, ut in Hyperbola *Apollonii.*

Series quidem, etiamſi quantitates quæ per eas quæruntur ſint finitæ magnitudinis, ſæpiſſime ſolent divergendo evadere infinite magnæ, in illis autem caſibus continuatæ in alteras partes nonnunquam convergunt & æquantur Radicibus quæſitis, vel ab iiſdem differunt quantitate determinata. Nonnunquam etiam utrinque continuatæ divergunt: ſæpius etiam ad utraſque partes nequeunt excurrere in infinitum propter Terminos impoſſibiles vel infinite parvos.

Inſuper ſicut Areæ Curvarum nunc augendæ nunc minuendæ ſunt datis quantitatibus, ut evadant veræ; ita etiam Summæ per hanc Propoſitionem inventæ nonnunquam differunt a veris, quo in caſu corrigendæ ſunt additione vel ablatione quantitatis datæ. Scilicet ubi Æquatio ad Summas talis eſt qualis efficit earum ultimam eſſe quantitatem magnitudinis finitæ vel infinite magnæ, ſemper opus eſt correctione: monſtrabo igitur in ſequente Propoſitione, qua ratione aſſumenda ſit Æquatio quæ ſemper efficiet ultimam eſſe nihil; eoque pacto ſumma inventa erit vera, nec augenda neque minuenda, ut hactenus monſtratum eſt.

PROPOSITIO VI.

Si Æquatio ad Summas ſit $S\times\overline{z^{\theta}+az^{\theta-1}+bz^{\theta-2}+\&c.} = mS'\times\overline{z^{\theta}+cz^{\theta-1}+dz^{\theta-2}+\&c.}$ *ultima Summarum erit finitæ magnitudinis in eo caſu ſolo, ubi eſt* $m=1$, *& ſimul* $a=c$.

Ad hanc Propoſitionem demonſtrandam, ſciendum eſt Summam S inveſtigari poſſe ex Æquatione definiente relationem inter eam ejuſque

valorem succeſſivum S′, eodem fere modo ac quantitas Fluxionalis ex ſua Æquatione. Illum in finem aſſumenda eſt pro S Series hujus formæ,

$$\frac{z^n}{p^z} \text{ in } A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \&c.$$

ubi n, p, A, B, C, D, &c. ſunt quantitates invariabiles. In caſu autem præſente ubi Summa quæſita eſt omnium ultima, adeoque ad diſtantiam infinitam remota; erit z etiam infinite magna, utpote quæ vel æqualis eſt illi diſtantiæ vel ab eadem differt quantitate finita: hac de cauſa Termini Seriei poſteriores ſunt infinite minores prioribus. Igitur ad abbreviandum computum, rejicio omnes poſt primum, utpote quæ in hac demonſtratione ſunt inutiles: ſic habeo $S = \frac{Az^n}{p^z}$; in qua ſcribendo S′ pro S, & $\overline{z+1}$ pro z; obtineo $S' = \frac{A\overline{z+1}|^n}{p^{z+1}}$. Hoſce valores ſubſtitue pro S & S′ in Æquatione ad Summas, vel quod eodem redit, in hac $S \times \overline{z+a} = mS \times \overline{z+c}$, membris reliquis ob rationes ſupra expoſitas neglectis; & reſultabit $\frac{Az^n}{p^z} \times \overline{z+a} = \frac{mA\overline{z+1}|^n}{p^{z+1}} \times \overline{z+c}$. Vel ducendo in p^{z+1}, & dividendo per A, $pz^n \times \overline{z+a} = m \times \overline{z+1}|^n \times \overline{z+c}$. Cæterum per Theorema *Newtoni* pro evolvendo Binomio, eſt $\overline{z+1}|^n = z^n + nz^{n-1}$, idque accurate propter z infinite magnam. Hunc valorem ſubſtitue, & evadet Æquatio $pz^n \times \overline{z+a} = m \times \overline{z^n + nz^{n-1}} \times \overline{z+c}$, quæ diviſa per z^{n-1}, fit $pz \times \overline{z+a} = m \times \overline{z+n} \times \overline{z+c}$, vel $pzz + paz = mzz + \overline{n+c} \times mz + mnc$; in qua conferendo membra homologa, erit $p = m$ & $pa = \overline{n+c} \times m$, unde $n = a - c$: & proinde ultima Summa S, quæ antea erat $\frac{Az^n}{p^z}$, jam evadit $\frac{Az^{a-c}}{m^z}$, ubi notandum eſt Coefficientem A non determinari. Fingatur nunc $m = 1$, & ſimul $a = c$; & Summa fiet $\frac{Az^0}{1^z}$: eſt vero $z^0 = 1$, & $1^z = 1$, etiamſi z ſit infinite magna; atque adeo ultima Summarum eſt finita, quippe æqualis quantitati A ubi eſt $m = 1$, & ſimul $a = c$: alius autem non datur caſus in quo $\frac{Az^{a-c}}{m^z}$ eſt quantitas finita ubi z eſt infinite magna. Conſtat igitur Propoſitio.

Corol-

COROLLARIUM.

Si m ſit minor unitate, ultima Summa erit infinite magna; & infinite parva ubi m eſt major unitate. Et ſi m ſit unitas, Summa illa erit infinite magna vel parva prout a eſt major aut minor c. Igitur in Æquatione ad Summas ſi m ſit major unitate, vel unitati æqualis & ſimul a minor c; ultima Summa ſemper erit nihil & nulla Correctione opus erit.

EXEMPLUM.

Sit Summa prima $A = 1$, ſecunda $B = \frac{9}{8}A$, tertia $C = \frac{25}{24}B$, quarta $D = \frac{49}{48}C$, quinta $E = \frac{81}{80}D$, &c. Atque Æquatio ad eaſdem erit $S \times \overline{zz + z + 1\frac{1}{4}} = S' \times \overline{zz + z}$; quæ collata cum Æquatione generali, dat $m = 1$, $a = 1$, $c = 1$; & inde $a - c = 0$; unde per hanc Propoſitionem ultima Summarum, id eſt factum ſub omnibus numeris $1 \times \frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{81}{80} \times \frac{121}{210} \times$ &c. in infinitum, eſt quantitas finita.

In Æquatione ad Summas ſubſtitue $S - T$ pro S, & invenies $S = -4T \times \overline{zz + z}$, quæ quoniam prodit negativa reſpectu ipſius T, exhibet Summam Terminorum non a dato Termino uſque in infinitum, ſed a dato Termino uſque ad principium Seriei.

Quod ut clarius pateat, proponantur duæ Æquationes $Szz = S' \times \overline{zz - 1}$, & $Sz = S'\overline{z + 1}$: utravis harum dabit eandem Æquationem ad Terminos, ſcilicet $Tz = T'\overline{z + 2}$. Ex priore tamen deducitur $S = -T \times \overline{zz + 1}$, & ex poſteriore $S = T\overline{z + 1}$. In primo caſu S eſt Summa Terminorum ab initio uſque ad T; & in ſecundo S eſt Summa ipſius T & omnium ſequentium in infinitum. In Æquatione ad Terminos ſcribantur numeri 1, 2, 3, 4, &c. ſucceſſive pro z, & uſurpetur $\frac{1}{2}$ pro primo Termino; atque prodibit Series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \text{\&c.}$$

Ubi ſi quæratur aggregatum primorum quatuor Terminorum, ſcribe 5 pro z in $T \times \overline{zz - 1}$ valore Summæ S priore, & Terminum quintum $\frac{1}{5.6}$ pro T, atque obtinebis $\frac{25 - 1}{5.6} = \frac{4}{5} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$, quatuor

quatuor utique Terminis initialibus. At si vis Summam omnium omnino Terminorum, præter illos quatuor, scribe 5 pro z, & $\frac{1}{5.6}$ pro T, in $T\overline{z+1}$ valore ipsius S posteriore; ac habebis $\frac{6}{5.6} = \frac{1}{5} = \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \frac{1}{8.9} +$ &c. Et hi duo valores ipsius S simul additi, hoc est, Summa numerorum $\frac{4}{5}$ & $\frac{1}{5}$ conficit $\frac{4+1}{5} = 1$ pro valore omnium omnino Terminorum ab initio usque ad infinitum.

Propositio VII.

Si Æquatio ad Seriem sit $\overline{z-n}\,T + \overline{m-1}.zT'$, *erit*

$$S = \frac{m-1}{m}T + \frac{n}{z}\times\frac{A}{m} + \frac{n+1}{z+1}\times\frac{B}{m} + \frac{n+2}{z+2}\times\frac{C}{m} + \frac{n+3}{z+3}\times\frac{D}{m} + \frac{n+4}{z+4}\times\frac{E}{m} + \text{\&c.}$$

Finge Summam S æqualem esse Termino T ducto in quantitatem y, hoc est $S = Ty$; dein scribe valores indeterminatarum posteriores $S-T$, T', y' pro prioribus S, T, & y respective, atque habebitur $S - T = T'y'$; quæ ablata de priore $S = Ty$, relinquit $T = Ty - T'y'$, unde est $T' = T\times\frac{y-1}{y'}$. Sed per Æquationem ad Seriem scilicet $\overline{z-n}\,T + \overline{m-1}.zT'$, est $T' = -\frac{\overline{z-n}\,T}{z.\overline{m-1}}$. Adeoque æquando sibi invicem duos ipsius T′ valores, erit $T\times\frac{y-1}{y'} = -\frac{\overline{z-n}\,T}{z.\overline{m-1}}$: quæ divisa per T, & ducta in $\overline{m-1}.y'$, evadit $\overline{m-1}.y - m + 1 = -y' + \frac{n}{z}y'$, vel $\overline{m-1}.y + y' - \frac{n}{z}y' - m + 1 = 0$: quæ est Æquatio Differentialis ex cujus resolutione dabitur Radix y. Illum in finem assume

$$y = a + \frac{b}{z} + \frac{c}{z.\overline{z+1}} + \frac{d}{z.\overline{z+1}.\overline{z+2}} + \frac{e}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \text{\&c.}$$

tum scribe pro y & z, valores earum succedaneos y' & $z+1$ respective, atque habebis

$y' = a +$
$$\frac{b}{z+1} + \frac{c}{z+1.z+2} + \frac{d}{z+1.z+2.z+3} + \frac{e}{z+1.z+2.z+3.z+4} + \&c.$$

hoc eſt, $y' = a + \frac{b}{z} + \frac{c-b}{z.z+1} + \frac{d-2c}{z.z+1.z+2} + \frac{e-3d}{z.z+1.z+2.z+3} + \&c.$

Atque prior valor ipſius y' multiplicatus per $\frac{n}{z}$, evadit

$$\frac{n}{z}y' = \frac{na}{z} + \frac{nb}{z.z+1} + \frac{nc}{z.z+1.z+2} + \frac{nd}{z.z+1.z+2.z+3} + \&c.$$

Hoſce valores ad eandem formam jam reductos ſubſtitue in Æquatione, & reſultabit

$$ma - m + 1 + \frac{mb - na}{z} + \frac{mc - \overline{n+1}.b}{z.z+1} + \frac{md - \overline{n+2}.c}{z.z+1.z+2} + \&c. = 0.$$

Ubi ponendo membra homologa æqualia nihilo, habebitur $a = \frac{m-1}{m}$, $b = \frac{n}{m}a$, $c = \frac{n+1}{m}b$, $d = \frac{n+2}{m}c$, $e = \frac{n+3}{m}d$, &c. Et hiſce datis, dabitur valor Radicis y, quæ denique ducta in T exhibebit pro S, Seriem in Propoſitione exhibitam. Q. E. D.

Corollarium. Si n ſit numerus integer & negativus, vel nihil; valor ipſius S abrumpet, exiſtente Serie ſummabili. Et ubi m eſt negativus, Series erit infinite magna. Hic autem excipio caſum in quo eſt $m=0$; tum enim Series ſummabitur per Exemplum primum Propoſitionis quintæ.

Exemplum I.

Proponatur Series ſummanda $\frac{1}{2} + \frac{1}{3}A + \frac{2}{5}B + \frac{3}{7}C + \frac{4}{9}D + \frac{5}{11}E + \&c.$ Æquatio eam definiens eſt $z - \frac{1}{2}T - 2zT' = 0$, ubi valores ſucceſſivi ipſius z ſunt $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. & illa collata cum Æquatione generali dat $n = \frac{1}{2}$, $m - 1 = -2$, ſive $m = -1$; quibus ſubſtitutis, oritur

$$S = 2T - \frac{A}{2z} - \frac{3B}{2z+2} - \frac{5C}{2z+4} - \frac{7D}{2z+6} - \frac{9E}{2z+8} - \&c.$$

In qua si substituatur quilibet Terminus pro T, & pro z valor suus correspondens, erit S Summa ipsius T & Terminorum omnium sequentium usque in infinitum. Colligo itaque duodecem Terminos initiales; eorumque aggregatum emergit .78533961813. Dein ut obtineam Summam reliquorum, scribo Terminum decimum tertium, id est, .0000302941I pro T, & pro z valorem suum debitum $13\frac{1}{2}$; atque habeo

$$S = .00006058822 - \frac{1}{27}A - \frac{3}{29}B - \frac{5}{31}C - \frac{7}{33}D - \frac{9}{35}E - \&c.$$

Ubi Termini prodeunt alternatim negativi & affirmativi, hosce dispono in una columna & illos in altera ut vides

.00006058822	.00000224401
23214	3744
794	204
61	20
7	3
1	1
+ .00006082899	− .00000228373

Tum Summam negativorum .00000228373 auferens ab ea affirmativorum .00006082899, habeo S = .00005854526, quæ adjecta Summæ initialium exhibet .78530816339 pro valore Seriei propositæ, id est, pro Area Circuli cujus Diamiter est unitas.

Exemplum II.

Quæratur valor Seriei $1 - \frac{1}{2}A - \frac{3}{4}B - \frac{5}{6}C - \frac{7}{8}D - \frac{9}{10}E - \&c.$

Æquatio definiens relationem Terminorum est $\overline{z - \frac{1}{2}}\,T + zT' = 0$, in qua valores Abscissæ z sunt 1, 2, 3, 4, &c. comparando autem hanc Æquationem cum illa in Theoremate, erit $n = \frac{1}{2}$, $m - 1 = 1$, & $m = 2$: atque exinde

$$S = \frac{1}{2}T + \frac{A}{4z} + \frac{3B}{4z+4} + \frac{5C}{4z+8} + \frac{7D}{4z+12} + \frac{9E}{4z+16} + \&c.$$

Collige

Collige nunc decem Terminos initiales Seriei transmutandæ, eorumque Summam reperies fore .6168670654. Deinde ut habeatur Summa reliquorum; in valore ipsius S scribe Terminum decimum primum, id est, .1761970520 pro T & 11 pro z; atque prodibit

$$S = .0880985260 + \frac{1}{44} A + \frac{3}{48} B + \frac{5}{52} C + \frac{7}{56} D + \frac{9}{60} E + \&c.$$

Et ineundo computum ut in margine, invenietur S = .0902397156, quæ addita aggregato initialium prius invento efficit .7071067810 pro valore Seriei, id est pro $\sqrt{\frac{1}{2}}$. Etenim $\sqrt{\frac{1}{2}}$ sic scripta $\overline{1+1}|^{-\frac{1}{2}}$, & evoluta per Theorema *Newtoni*, evadit Series de qua nunc egimus.

.0880985260
20022392
1251400
120327
15041
2256
388
74
15
3
S = .0902397156

SCHOLION.

Omnis Series cujus Termini sunt per vices negativi & affirmativi, si transmutetur per hanc Propositionem, migrabit in aliam celerius convergentem cujus Termini sunt ejusdem signi. Et è contra omnis Series cujus Termini sunt ejusdem signi, abibit in aliam cujus Termini sunt alternatim negativi & affirmativi; quæ tamen non converget celerius priore præterquam ubi Transmutatio inchoatur a Terminis ab initio satis remotis. Et si Series transmutetur, & rursus transmutetur ea quæ ex prima Transmutatione prodiit, emerget ea prius proposita. Exempli gratia si Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$ transmutetur, prodibit $\frac{1}{2} + \frac{1}{3} A + \frac{2}{5} B + \frac{3}{7} C + \frac{4}{9} D + \&c.$ dein si novissima transmutetur emerget prima $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$ hoc est, transfert Seriem de Tangente ad Sinum, vel de Sinu ad Tangentem. Sed in hisce casibus debet operatio ordiri ad primum Terminum Seriei, nempe ubi agitur de Transmutatione totius.

PROPO-

PROPOSITIO VIII.

Si Æquatio ad Seriem ſit $T' = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, *&* *ponatur* $A = \overline{z-n}\, T$, $B = \frac{n}{z} A$, $C = \frac{n+1}{z+1} B$, $D = \frac{n+2}{z+2} C$, $E = \frac{n+3}{z+3} D$, *&c. Erit* $S = \frac{A}{m} + \frac{B}{m+1} + \frac{C}{m+2} + \frac{D}{m+3}$ *&c.*

Etenim ſuppone $S = T \times \overline{z-n} \times y$, & per methodum Differentialem erit $S - T = T' \times \overline{z-n+1} \times y'$, harum Æquationum Differentia dabit $T = T \times \overline{z-n} \times y - T' \times \overline{z-n+1} \times y'$. Pro T' ſcribe ſuum valorem $\frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, atque habebitur $T = T \times \overline{z-n} \times y - T' \times \frac{z-m}{z} \times \overline{z-n} \times y'$ quæ diviſa per $T \times \overline{z-n}$, evadit $\frac{1}{z-n} = y - \frac{z-m}{z} y'$, vel $y - y' + \frac{m}{z} y' - \frac{1}{z-n} = 0$. Ex hac vero Æquatione eruetur Radix y ad modum ſequentem. Finge

$$y = \frac{a}{m} + \frac{b}{\overline{m+1}.z} + \frac{c}{\overline{m+2}.z.\overline{z+1}} + \frac{d}{\overline{m+3}.z.\overline{z+1}.\overline{z+2}} + \&c.$$

Erit $y' =$

$$\frac{a}{m} + \frac{b}{\overline{m+1}.\overline{z+1}} + \frac{c}{\overline{m+2}.\overline{z+1}.\overline{z+2}} + \frac{d}{\overline{m+3}.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Et $y - y' =$

$$\frac{b}{\overline{m+1}.z.\overline{z+1}} + \frac{2c}{\overline{m+2}.z.\overline{z+1}.\overline{z+2}} + \frac{3d}{\overline{m+3}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Atque $\frac{m}{z} y' =$

$$\frac{am}{mz} + \frac{bm}{\overline{m+1}.z.\overline{z+1}} + \frac{cm}{\overline{m+2}.z.\overline{z+1}.\overline{z+2}} + \frac{dm}{\overline{m+3}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} \&c.$$

Adeoque

Adeoque $y - y' + \frac{m}{z}y' =$

$$\frac{a}{z} + \frac{c}{z.\overline{z+1}} + \frac{b}{z.\overline{z+1}.\overline{z+2}} + \frac{d}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Eſt vero $\frac{1}{z-n} =$

$$\frac{1}{z} + \frac{n}{z.\overline{z+1}} + \frac{n.\overline{n+1}}{z.\overline{z+1}.\overline{z+2}} + \frac{n.\overline{n+1}.\overline{n+2}}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Igitur $y - y' + \frac{m}{z}y - \frac{1}{z-n} =$

$$\frac{a-1}{z} + \frac{b-n}{z.\overline{z+1}} + \frac{c-n.\overline{n+1}}{z.\overline{z+1}.\overline{z+2}} + \frac{d-n.\overline{n+1}.\overline{n+2}}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c. = 0.$$

Pone jam Numeratores nihil, ut evaneſcant Termini; & pro determinatione Coefficientium habebis Æquationes ſequentes $a=1$, $b=n$, $c=n.\overline{n+1}$, $d=n.\overline{n+1}.\overline{n+2}$, &c. ſubſtitue hoſce valores in Serie aſſumpta pro y, vice ipſarum a, b, c, d, &c. & valor ipſius y, exinde ortus ductus in $\overline{z-n}\,T$, exhibebit pro S, Seriem in Theoremate poſitam. Q. E. D.

Corollarium. Si n ſit integer & negativus, vel nihil; Series ſummabitur accurate per hoc Theorema. Et ſi m ſit nihil vel negativus, Series erit infinite magna. Inſervit autem hæc Propoſitio & ſuperior Quadraturæ Curvarum binomialium. Hæc uſui venit ubi in Ordinata $x^{\theta} \times \overline{e+fx^{\eta}}|^{\lambda}$, eſt $e+fx^{\eta}=0$; ſuperior vero adhibenda eſt ubi contrarium accidit.

Exemplum I.

Quæratur valor Seriei $1 + \frac{1.1}{2.3}A + \frac{3.3}{4.5}B + \frac{5.5}{6.7}C + \frac{7.7}{8.9}D + \frac{9.9}{10.11}E + \&c.$

Æquatio eam definiens eſt $T' = \frac{z-\frac{1}{2}}{z} \times \frac{z-\frac{1}{2}}{z+\frac{1}{2}}T$, uti patebit ſcribendo valores 1, 2, 3, 4, &c. ſucceſſive pro z. Æquatio autem in Theoremate comparata cum hac, dat $m=\frac{1}{2}$, $n=\frac{1}{2}$; unde $A=\overline{z-\frac{1}{2}}\,T$, $B=$

$\frac{A}{2z}$, $C=\frac{3B}{2z+2}$, $D=\frac{5C}{2z+4}$, $E=\frac{7D}{2z+6}$, &c. atque $S=\frac{2}{1}A+\frac{2}{3}B+\frac{2}{5}C+\frac{2}{7}D+\frac{2}{9}E+$ &c. Ineundo computum reperio aggregatum duodecem Terminorum eſſe 1.407397508. Dein ſubſtituo Terminum decimum tertium, id eſt, $\frac{1}{25}\times.161180258$ pro T, & pro z valorem ſuum correſpondentem 13, ſic prodit

$A=\frac{1}{2}\times.161180258,$	.166180258
$B=\frac{1}{2}\times$ - 6199241,	2066414
$C=\frac{1}{2}\times$ - - 664204,	132841
$D=\frac{1}{2}\times$ - - 110701,	15814
$E=\frac{1}{2}\times$ - - - 24261,	2691
$F=\frac{1}{2}\times$ - - - 6410,	583
$G=\frac{1}{2}\times$ - - - 1959,	151
$H=\frac{1}{2}\times$ - - - 670,	45
$I=\frac{1}{2}\times$ - - - - 250,	15
$K=\frac{1}{2}\times$ - - - - 102,	5
$L=\frac{1}{2}\times$ - - - - 44,	2
$M=\frac{1}{2}\times$ - - - - 20,	1
	$S=.163398820$

Ex hoc computo habeo $S=.163398820$, valorem utique Terminorum omnium poſt decimum ſecundum; quem itaque adjiciens Summæ initialium, 1.407397508, habeo 1.570796328 pro valore Seriei ſummandæ, id eſt, pro longitudine Arcus ſemicircularis cujus Diameter eſt unitas.

Termini initiales, quando eorum Summa eruitur, facillime reducuntur in fractiones decimales per regulam ſequentem, pone $a=1$, $b=a-\frac{1}{2}a$, $c=b-\frac{1}{4}b$, $d=c-\frac{1}{6}c$, $e=d-\frac{1}{8}d$, &c. eruntque Termini a, $\frac{1}{3}b$, $\frac{1}{5}c$, $\frac{1}{7}d$, $\frac{1}{9}e$, &c.

Exemplum II.

Proponatur Series *Brounkeri* $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+$ &c. Æquatio

quatio ad eandem eſt $T' = \frac{z-1}{z} \times \frac{z-\frac{1}{2}}{z+\frac{1}{2}} T$, vel etiam $T' = \frac{z-1}{z} \times \frac{z-\frac{3}{2}}{z-\frac{1}{2}} T$, ſcilicet ſumendo initium Abſciſſæ z a diverſis punctis. In priore valores ipſius z ſunt $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, &c. $m=1$, $n=\frac{1}{2}$, $A=\overline{z-\frac{1}{2}}T$, $B=\frac{A}{2z}$, $C=\frac{3B}{2z+2}$, $D=\frac{5C}{2z+4}$, &c. In poſteriore valores ipſius z ſunt 2, 3, 4, 5, &c. $m=1$, $n=\frac{3}{2}$, $A=\overline{z-\frac{3}{2}}T$, $B=\frac{3A}{2z}$, $C=\frac{5B}{2z+2}$, $D=\frac{7C}{2z+4}$, &c. & in utroque erit $S=A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ &c. Dabitur ergo S ad plures figuras ſumendo pro T quemvis è Terminis ſat longe diſtantibus a principio. Et ſimiliter in aliis caſibus inſtituere licet computum, idque duobus modis diverſis ubi Termini ſunt aſſignabiles. Hæc autem Series hunc in modum ſcripta $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-$ &c. facilius tractabitur per Propoſitionem tertiam aut ſeptimam.

Scholion.

Hactenus egimus de Summatione Serierum quæ prodeunt in Quadratura Curvarum binomialium, & ſimilibus. Cæterum procedere licet eodem modo in caſibus difficilioribus: nam Summa Seriei determinatur & erui poteſt ex data relatione Terminorum; idque reſolvendo Æquationem Differentialem ut in noviſſimis duabus Propoſitionibus. Problema autem eſſet laborioſum quærere Summas independenter a Terminis, quando Termini non ſunt aſſignabiles; ideoque quæſivi quantitatem quæ ducta in Terminum T exhibet Summam S. Sic etiam Areæ Curvarum facilius eruuntur mediantibus Ordinatis, nam Series hac ratione prodeuntes ſunt ſimpliciſſimæ. Et hiſce præmiſſis pergo ad methodum reſolvendi Radices Æquationum Differentialium in Series magis quidem compoſitas, at ſimul longe magis convergentes quam ſuperiores; quas ea tantùm de cauſa exhibui, quod eſſent ſimplices & uſibus vulgaribus ſufficientes.

Hucuſque Summam quamlibet deſignavimus per S, ejuſque Terminos per T, T′, T″, &c. At in ſequentibus etiam deſignabimus Series vel Summas, per S2, S3, S4, &c. earumque Terminos per T2, T′2, T″2,

$T''2$, &c. $T3$, $T'3$, $T''3$, &c. & ſic porro: adeo ut ſit

$$S2 = T2 + T'2 + T''2 + \&c.$$
$$S3 = T3 + T'3 + T''3 + \&c.$$
$$S4 = T4 + T'4 + T''4 + \&c.$$
$$\&c.$$

Et ſicut in Serie S, Summæ ſucceſſivæ deſignantur per S', S'', &c. ita in Serie $S2$, denotabuntur per $S'2$, $S''2$, &c. atque in Serie $S3$ per $S'3$, $S''3$, &c. Et ſic in cæteris. Hanc autem notationem introducere coactus ſum, propterea quod Summæ & Termini diverſarum Serierum ſimul conſiderandæ veniunt in eadem Æquatione.

Propositio IX.

Data relatione inter duas Summas in diverſis Seriebus, & Æquatione ad Terminos in alterutra; invenire Æquationem ad Terminos in altera.

Problema ſolvitur progrediendo ex relatione variabilium præſente ad ſuccedentem, ut exinde eliminentur Summæ; id quod Exemplis patebit.

Exemplum I.

Sint S & $S2$ duæ Summæ in diverſis Seriebus, earumque relatio $S = \frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}} T + S2$; ſit vero $T' = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, relatio Terminorum Seriei S: atque ex hiſce datis opporteat eruere Æquationem ad Terminos Seriei $S2$. In Æquatione $S = \frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}} T + S2$, exhibente relationem Summarum, ſubſtitue valores variabilium conſequentes pro antecedentibus, hoc eſt, S' ſive $S - T$ pro S; $S'2$, ſive $S2 - T2$ pro $S2$; T' pro T, & $z+1$ pro z; atque habebis $S - T = \frac{\overline{m+1}.\overline{z+1-n}}{m.\overline{m+1}} T' + S2 - T2$, quæ ablata de priore relinquit Æquationem liberatam a Summis, ſcilicet $T = \frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}} T - \frac{\overline{m+1}.\overline{z+1-n}}{m.\overline{m+1}} T' + T2$; unde eſt $T2 = \frac{\overline{m+1}.\overline{z+1-n}}{m.\overline{m+1}} T' - \frac{\overline{m+1}.\overline{z-m-n}}{m.\overline{m+1}} T$: in qua

ſubſti-

ſubſtitue pro T′ valorem proprium $\frac{z-m}{z}\times\frac{z-n}{z-n+1}$ T, atque obtinebis $T2=\frac{n.\overline{m-n+1}}{\overline{m+1}.z.\overline{z-n+1}}$ T. In hoc inſuper valore ſcribe valores indeterminatarum ſuccedentes T′2, T′, & $z+1$ pro præſentibus T2, T, & z, & orietur $T'2=\frac{n.\overline{m-n+1}}{\overline{m+1}.\overline{z+1}.\overline{z-n+2}}$ T′; ubi pro T′ rurſus ſubſtituendo ſuum valorem, emergit $T'2=\frac{n.\overline{m-n+1}}{\overline{m+1}.\overline{z+1}.\overline{z-n+2}}T\times\frac{z-m}{z}\times\frac{z-n}{z-n+1}$, ex qua prodit $T=\frac{\overline{m+1}.z.\overline{z+1}.\overline{z-n+1}.\overline{z-n+2}}{n.\overline{m-n+1}.\overline{z-m}.\overline{z-n}}$ T′2. Æquatio autem prius inventa $T2=\frac{n.\overline{m-n+1}}{\overline{m+1}.z.\overline{z-n+1}}$ T dat $T=\frac{\overline{m+1}.z.\overline{z-n+1}}{n.\overline{m-n+1}}$ T2. Æquentur jam ſibi invicem duo valores Termini T, & incides in Æquationem $T'2=\frac{z-m}{z+1}\times\frac{z-n}{z-n+2}$ T2, quæ exprimit relationem Terminorum Seriei S2. Q. E. I.

Exemplum II.

Sit jam relatio inter Summas $S=\frac{3}{4}\times\frac{4z+3}{4z+2}T+S2$, & $zT+3T'\times\overline{z+1}$ Æquatio ad Seriem S. In Æquatione ad Summas ſcribe valores variabilium ſucceſſivos, & habebitur $S-T=\frac{3}{4}\times\frac{4z+7}{4z+6}T'+S2-T2$: quam de priore auferens habebis $T=\frac{3}{4}\times\frac{4z+3}{4z+2}T-\frac{3}{4}\times\frac{4z+7}{4z+6}T'+T2$; ex qua comperies $T2=\frac{1}{4}\times\frac{4z-1}{4z+2}T+\frac{3}{4}\times\frac{4z+7}{4z+6}T'$. Cæterum per Æquationem ad Seriem S, eſt $3T'=-\frac{z}{z+1}T$, quo ſcripto, prodit $T2=+\frac{1}{4}\times\frac{4z-1}{4z+2}T+\frac{1}{4}\times\frac{4z+7}{4z+6}\times\frac{-z}{z+1}T$, vel $T2=-\frac{3}{4}T\times\frac{1}{2z+1}\times\frac{1}{2z+2}\times\frac{1}{2z+3}$. Atque rurſus

ſus

ſus progrediendo ad valores indeterminatarum ſuccedentes, habebitur $T'2 = -\frac{3}{4} T' \times \frac{1}{2z+3} \times \frac{1}{2z+4} \times \frac{1}{2z+5}$, vel ſubſtituendo $-\frac{z}{z+1} T$ pro $3T'$, $T'2 = \frac{1}{2} \times \frac{1}{2z+2} \times \frac{1}{2z+3} \times \frac{1}{2z+4} \times \frac{1}{2z+5} T$: unde $T = 2.\overline{2z+2}.\overline{2z+3}.\overline{2z+4}.\overline{2z+5}.\ T'2$; atque per valorem ipſius $T2$ eſt idem $T = -\frac{4}{3} \times \overline{2z+1} \times \overline{2z+2} \times \overline{2z+3}\ T2$; quibus duobus valoribus inter ſe æquatis, obtinebitur $\overline{zz + \frac{1}{2}z} \times T2 + \overline{zz + \frac{9}{2}z + 5} \times 3T'2 = 0$. Æquatio utique ad Terminos Seriei S. Et in aliis caſibus ſimiliter procedendum eſt.

SCHOLION.

Per hanc Propoſitionem Series infinitæ inter ſe comparari poſſunt. Nam Æquatio exhibens relationem inter Summas S & S2, dabit unam ex data altera: & Termini ipſarum S & S2, dabuntur ex ſuis Æquationibus, quarum una aſſumitur, altera vero ex ea aſſumpta eruitur: ut in Exemplis hujus Propoſitionis. Ut ſi relatio Summarum ſit $S = \frac{2z-1}{2} T + S2$, & Æquatio ad Seriem S, $T' = \frac{zz-2z+1}{zz} T$; invenies $T2 = \frac{T}{2zz}$, & $T'2 = \frac{zz-2z+1}{zz+2z+1} T2$ Æquationem ad Seriem S2. Sint jam 2, 3, 4, 5, &c. valores ſuccedentes indeterminatæ z; & uſurpando unitatem pro T invenies

$$S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \&c.$$

$$S2 = \frac{1}{8} \text{ in } 1 + \frac{1}{9} + \frac{1}{36} + \frac{1}{100} + \frac{1}{225} + \&c.$$

In priore Denominatores ſunt quadrata numerorum naturalium, & in poſteriore quadrata numerorum Triangularium. Relatio autem inter Series $S = \frac{2z-1}{2} T + S2$, ſcribendo 2 pro z, & unitatem pro T, dabit $S2 = S - \frac{3}{2}$, & ducendo in 8, erit $8S - 12 = 8S2 = 1 + \frac{1}{9} +$

$\frac{1}{36}$

$\frac{1}{36}+\frac{1}{100}+\frac{1}{225}+$ &c. Et similiter cum Serie in qua Denominatores sunt quadrata numerorum naturalium comparari possunt eæ ubi Denominatores sunt quadrata Pyramidalium, Triangulo-Triangularium, &c. Atque in genere possunt comparari omnes Series quæ definiuntur Æquationibus sequentibus

$$T'=\frac{zz+a}{z+b\,.\,z+c}T,\ T'=\frac{zz+a}{z+b+1\,.\,z+c+1}T,\ T'=\frac{zz+a}{z+b+2\,.\,z+c+2}T, \text{ \&c.}$$

Ubi numerator manet idem, Denominatores autem formantur scribendo continue $z+1$ pro z.

Propositio X.

Invenire valorem Seriei quam proxime, quæ definitur Æquatione hujus formæ

$$T\times\overline{z^\theta+az^{\theta-1}+bz^{\theta-2}+\&c.}=rT'\times\overline{z^\theta+cz^{\theta-1}+dz^{\theta-2}+\&c.}$$

Casus I.

Primo sit $r=1$, & fingamus Æquationem ad Seriem esse $T'=\frac{z-m}{z}\times\frac{z-n+1}{z-n}T$. Ponamus esse $S=\frac{z+p}{q}T$ quam proxime, vel $S=\frac{z+p}{q}T+S2$ accurate: & progrediendo ad valores indeterminatarum succedaneos, erit $S-T=\frac{z+p+1}{q}T'+S2-T2$; harum Æquationum differentia dabit $T=\frac{z+p}{q}T-\frac{z+p+1}{q}T'+T2$; unde $T2=\frac{z+p+1}{q}T'-\frac{z+p-q}{q}T$. Pro T' substitue suum valorem $\frac{z-m}{z}\times\frac{z-n}{z-n+1}T$, atque proveniet $T2=\frac{z+p+1}{q}\times\frac{z-m}{z}\times\frac{z-n}{z-n+1}T-\frac{z+p-q}{q}T$: qui reductus ad communem Denominatorem dat

$T2=$

$$T2 = \frac{\overline{q-m}\times zz \left.\begin{matrix} +\overline{m-q}\times\overline{n-1} \\ -\overline{n-p}\times\overline{m+1}\end{matrix}\right\} z + m.n.\overline{p+1}}{q.z.\overline{z-n+1}}\, T,$$

Jam quo minor eſt Summa S2, eo proprius quantitas $\frac{z+p}{q}$T accedet ad valorem ipſius S; & eo minor erit S2, quo minor eſt Terminus ejus primus T2: is vero evadet minimus ubi variabilis z eſt minimarum dimenſionum in Numeratore ejus valoris; nam hic z magna ſupponitur: ponantur ergo Coefficientes dignitatum z^2 & z nihilo æquales, & habebuntur duæ Æquationes $q-m=0$, atque $\overline{m-q}\times\overline{n-1}-\overline{n-p}\times\overline{m+1}=0$, ad determinandum duas quantitates aſſumptas p, q. Prima dat $q=m$, ex quâ & ſecundâ eruitur $p=-\frac{n}{m+1}$. Unde erit $T2=\frac{n.\overline{m-n+1}}{\overline{m+1}.z.\overline{z-n+1}}T$, atque fere $S=\frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}}T$. Q. E. I.

Et eadem prorſus ratione invenietur approximatio, ubi Æquatio ad Seriem eſt $T'=\frac{zz+az+b}{zz+cz+d}T$, vel magis compoſita; ſed Coefficientes poſt b, & d in hoc computo non ingrediuntur.

CASUS II.

Sit jam Æquatio $T\times\overline{zz+az+b}+rT'\times\overline{zz+cz+d}=0$, exiſtente r numero quolibet, exceptâ unitate. Finge $S=p\times\frac{z+m}{z+n}T+S2$; dein ſcribe valores variabilium poſteriores pro prioribus, & emerget $S-T=p\times\frac{z+m+1}{z+n+1}T'+S2-T2$, quæ ablata de priore relinquit $T=p\times\frac{z+m}{z+n}T-p\times\frac{z+m+1}{z+n+1}T'+T2$: hæc Æquatio dabit $T2=\frac{z-pz-mp+n}{z+n}T+p\times\frac{z+m+1}{z+n+1}T'$: ſed per Æquationem ad Seriem S, eſt $T'=-\frac{T}{r}\times\frac{zz+az+b}{zz+cz+d}$; quo ſcripto pro T', prodit

T2 =

$T2 = \frac{z - pz - mp + n}{z + n} T - \frac{pT}{r} \times \frac{z + m + 1}{z + n + 1} \times \frac{zz + az + b}{zz + cz + d}$. Si hujus valoris membra reducantur ad communem Denominatorem, & ponantur Coefficientes trium altissimarum dignitatum ipsius z æquales nihil; Æquatio prima dabit $rp + p = r$, secunda $\overline{m - n} \times \overline{r + 1} = c - a$, tertia denique $\overline{c - a} \times \overline{2n + 1} + d - b = \overline{m - n} \times \overline{rc + rn + r + n} + ma$: & hæ tres Æquationes dant $p = \frac{r}{r + 1}$, $m = c - \frac{b - d}{a - c} - \frac{1}{r + 1}$, & $n = m + \frac{a - c}{r + 1}$. Dantur itaque Quantitates assumptæ p, m, & n; atque exinde etiam Quantitas $p \times \frac{z + m}{z + n} T$ quæ proxime æqualis est Seriei S. Q. E. I.

PROPOSITIO XI.

Si $T' = \frac{z - m}{z} \times \frac{z - n}{z - n + 1} T$ *sit Æquatio exhibens relationem Terminorum Seriei* S, *pone*

$$T2 = \frac{1}{m} \times \frac{m}{m + 1} \times \frac{n}{z} \times \frac{m - n + 1}{z - n + 1} T,$$

$$T3 = \frac{2}{m + 2} \times \frac{m + 1}{m + 3} \times \frac{n + 1}{z + 1} \times \frac{m - n + 2}{z - n + 2} T2,$$

$$T4 = \frac{3}{m + 4} \times \frac{m + 2}{m + 5} \times \frac{n + 2}{z + 2} \times \frac{m - n + 3}{z - n + 3} T3,$$

$$T5 = \frac{4}{m + 6} \times \frac{m + 3}{m + 7} \times \frac{n + 3}{z + 3} \times \frac{m - n + 4}{z - n + 4} T4,$$

$$T6 = \frac{5}{m + 8} \times \frac{m + 4}{m + 9} \times \frac{n + 4}{z + 4} \times \frac{m - n + 5}{z - n + 5} T5,$$

&c.

Eritque $S = \frac{\overline{m + 1} . z - 1 . n}{m . \overline{m + 1}} T +$

$$\frac{\overline{m + 3} . \overline{z + 2} - 2 . \overline{n + 1}}{\overline{m + 2} . \overline{m + 3}} T2 +$$

$$\frac{\overline{m + 5} . \overline{z + 4} - 3 . \overline{n + 2}}{\overline{m + 4} . \overline{m + 5}} T3 +$$

$$\frac{\overline{m + 7} . \overline{z + 6} - 4 . \overline{n + 3}}{\overline{m + 6} . \overline{m + 7}} T4 +$$

$$\frac{\overline{m + 9} . \overline{z + 8} - 5 . \overline{n + 4}}{\overline{m + 8} . \overline{m + 9}} T5 +$$

&c.

Per Propositionem præcedentem quantitas $\frac{\overline{m+1}\,\overline{z-n}}{m.\overline{m+1}}T$ proxime æqualis est Seriei S. Sit ergo $S = \frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}}T + S_2$ accurate, & invenies $T_2 = \frac{1}{m} \times \frac{m}{m+1} \times \frac{n}{z} \times \frac{m-n+1}{z-n+1}T$, atque $T'_2 = \frac{z-m}{z+1} \times \frac{z-n}{z-n+2}T_2$ Æquationem ad Terminos Seriei S_2: ex qua data, per Propositionem priorem eruitur quantitas $\frac{\overline{m+3}.\overline{z+2}-2\overline{n+1}}{m+2.\overline{m+3}}T_2$ proxime æqualis Seriei S_2. Dein assumendo $S_2 = \frac{\overline{m+3}.\overline{z+2}-2.\overline{n+1}}{m+2.\overline{m+3}}T_2 + S_3$, invenietur $T_3 = \frac{2}{m+2} \times \frac{m+1}{m+3} \times \frac{n+1}{z+1} \times \frac{m-n+1}{z-n+1}T_2$; & $T'_3 = \frac{z-m}{z+2} \times \frac{z-n}{z-n+3}T_3$ Æquatio ad Terminos Seriei S_3: unde comperies quantitatem $\frac{\overline{m+5}.\overline{z+4}-3.\overline{n+2}}{m+4.\overline{m+5}}T_3$ proxime æqualem Seriei S_3. Et sic porro proceditur. Igitur primus Terminus in valore ipsius S proxime æquatur ipsi S; atque Terminus secundus proxime æquatur ipsi S_2; tertius vero proxime æquatur ipsi S_3; & ita in reliquis: hoc est, Terminus primus proxime æqualis est Seriei cujus valor quæritur; secundus proxime æqualis est defectui primi Termini a vero; tertius proxime æqualis est defectui duorum primorum a vero; quartus proxime æqualis est defectui trium primorum a vero; & sic deinceps. Igitur valor Summæ S est verus & celerrime convergens. Q. E. D.

Corollarium. Abrumpet valor Seriei S hic exhibitus, ubi n vel $m-n+1$ est nihil aut numerus integer & negativus: atque in aliis casibus abibit in infinitum ad veritatem quam celerrime accelerans præterquam ubi, propter m nihil aut negativam, valor Seriei est infinite magnus.

Exemplum I.

Quæratur valor Seriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} +$ &c. Æquatio definiens relationem Terminorum est $T' = \frac{z-1}{z} \times \frac{z-1}{z}T$, existentibus

tibus 2, 3, 4, 5, &c. valoribus indeterminatæ z. Hæc autem collata cum Æquatione in Theoremate dat $m=1$, $n=1$; quibus ſcriptis prodeunt

$$T2=\frac{T}{2z^2},\ T3=\frac{8T2}{6.\overline{z+1}^2},\ T4=\frac{27T3}{10.\overline{z+2}^2},\ T5=\frac{64T4}{14.\overline{z+3}^2},\ \&c.$$

$$\text{Et } S=\frac{2z-1}{2}T+\frac{2z+2}{6}T2+\frac{2z+5}{10}T3+\frac{2z+8}{14}T4+\frac{2z+11}{18}T5 + \&c.$$

Collige jam decem Terminos initiales, eorumque aggregatum reperies 1.5497.6773.1166.5406.9. Tum pro z ſcribe ejus valorem decimum primum, ſive 12; & Terminum decimum primum $\frac{1}{121}$ pro T; atque per computum invenies

T =	.0082.6446.2809.9173.55	.0950.4132.2314.0495.8
T2 =	- - - 2869.6051.4233.24	1.2434.9556.1677.4
T3 =	- - - - 22.6398.8277.97	65.6556.6006.1
T4 =	- - - - - 3118.7593.62	7128.5928.3
T5 =	- - - - - - - 63.3652.70	123.2102.5
T6 =	- - - - - - - - 1.7188.93	2.9690.0
T7 =	- - - - - - - - - 583.96	920.9
T8 =	- - - - - - - - - - 23.78	34.9
T9 =	- - - - - - - - - - - 1.12	1.5
		S = .0951.6633.5681.6857.4

Ex hoc computo habemus S = .0951.6633.5681.6857.4, quæ adjecta aggregato initialium, exhibet 1.6449.3406.6848.2264.3 pro valore Seriei propoſitæ.

Exemplum II.

Summanda ſit Series $1+\frac{1.1}{2.3}A+\frac{3.3}{4.5}B+\frac{5.5}{6.7}C+\frac{7.7}{8.9}D+\frac{9.9}{10.11}E+\&c.$

Æquatio eandem denotans eſt $T'=\frac{z-\frac{1}{2}}{z}\times\frac{z-\frac{1}{2}}{z+\frac{1}{2}}T$, uti patebit ſcribendo valores 1, 2, 3, 4, &c. ſucceſſive pro z. Æquatio autem in Theoremate collata cum hac, dat $m=\frac{1}{2}$, $n=\frac{1}{2}$, adeoque

T2 =

$$T2 = \frac{2.1.1}{1.3.z.2z+1}T,\ T3 = \frac{2.4.9}{5.7.z+1.2z+3}T2,\ T4 = \frac{2.9.25}{9.11.z+2.2z+5}T3,$$
&c.

$$\text{Et } S = \frac{6.z - 1.2}{1.3}T + \frac{14.\overline{z+2} - 3.4}{5.7}T2 + \frac{22.\overline{z+4} - 5.6}{9.11}T3 + \&c.$$

Per additionem reperies Summam decem Terminorum initialium esse 1.3916.94645943.2880.5; dein ut eruatur Summa reliquorum scribe 11 pro z, & Terminum decimum primum pro T, & habebis

T =	.0083.9003.5809.6168.15	.1789.9383.0605.1587.3
T2 =	2210.8921.7644.71	1.0738.6191.4274.3
T3 =	15.1604.0349.56	45.9406.1665.3
T4 =	1963.2742.16	4570.9051.0
T5 =	38.8835.92	76.0818.3
T6 =	1.0484.94	1.8104.4
T7 =	358.18	562.5
T8 =	14.77	21.5
T9 =	71	1.0
		S = .1791.0168.0851.6085.6

Addatur nunc S aggregato initialium, & prodibit 1.5707.9632.6794.8966.1 pro valore Seriei, id est, pro semiperiferia Circuli, cujus Diameter est unitas.

Exemplum III.

Proponatur jam Series $\frac{1}{3} + \frac{1.3}{2.7}A + \frac{3.7}{4.11}B + \frac{5.11}{6.15}C + \frac{7.15}{8.19}D$ &c. quæ definitur Æquatione $T' = \frac{z-\frac{1}{2}}{z} \times \frac{z-\frac{1}{4}}{z+\frac{3}{4}}T$, in qua 1, 2, 3, 4, &c. sunt valores indeterminatæ z. Et erit $m = \frac{1}{2}$, $n = \frac{1}{4}$; & propterea

$$T2 = \frac{5.1.1}{6.z.4z+3}T,\ T3 = \frac{9.2.3}{14.z+1.4z+7}T2,\ T4 = \frac{13.3.5}{22.z+2.4z+11}T3,$$
&c.

$$\text{Et } S = \frac{6.z - 1.1}{1.3}T + \frac{14.\overline{z+2} - 2.5}{5.7}T2 + \frac{22.\overline{z+4} - 3.9}{9.11}T3 + \&c.$$

Colligo

Colligo jam novem Terminos initiales, quorum Summa prodit .5055.0041.4718.3195.8, & ſcribens 10 pro *z*, decimum vero Terminum pro T, habeo

T =	.0047.5565.5924.4791.67	.0935.2789.9848.0902.9
T2 =	921.6387.4505.41	4160.5406.2053.0
T3 =	6.8760.0058.48	19.5167.2893.3
T4 =	995.8557.00	2185.7755.9
T5 =	22.0991.75	40.9826.8
T6 =	6653.41	1.0937.5
T7 =	252.54	379.0
T8 =	11.51	16.1
T9 =	61	8
	S =	.0935.6970.2649.4765.3

Summa denique initialium addita ipſi S, conficit .5990.7011.7367.7961.1 pro valore Seriei, id eſt, pro Ordinata Curvæ Elaſticæ. Et hunc numerum *Jacobus Bernoullius* recte invenit contineri intra limites .5983 & .6004.

Exemplum IV.

Proponatur Series $1 + \frac{1.1}{2.5}A + \frac{3.5}{4.9}B + \frac{5.9}{6.13}C + \frac{7.13}{8.17}D + \frac{9.17}{10.21}E + \&c.$ quæ definitur Æquatione $T' = \frac{z-\frac{1}{2}}{z} \times \frac{z-\frac{3}{4}}{z+\frac{1}{4}}T$, exiſtentibus 1, 2, 3 4, &c. valoribus indeterminatæ ſuccedentibus. Eſt vero $m = \frac{1}{2}$, $n = \frac{3}{4}$; & proinde

$$T2 = \frac{1.1.3}{2.z.\overline{4z+1}}T,\ T3 = \frac{3.2.7}{10.\overline{z+1}.\overline{4z+5}}T2,\ T4 = \frac{5.3.11}{18.\overline{z+2}.\overline{4z+9}}T3,\ \&c.$$

$$\text{Et } S = \frac{2z-1}{1}T + \frac{2z+2}{5}T2 + \frac{2z+5}{9}T3 + \frac{2z+8}{13}T4 + \frac{2z+11}{17}T5 + \&c.$$

Summa novem Terminorum initialium eſt 1.2157.0599.7306.1360.6. Et ut habeatur Summa reliquorum, pone 10 pro *z*, & decimum Terminum pro T, ac per computum obtinebis

T =	.0050.1271.8406.8834.46	.0952.4164.9730.7854.7
T2 =	- - - 1833.9213.6837.20	8069.2540.2083.7
T3 =	- - - - 15.5605.4494.38	43.2237.3595.5
T4 =	- - - - - 2425.8219.16	5224.8472.0
T5 =	- - - - - - - 56.8742.44	103.7118.6
T6 =	- - - - - - - - - 1.7922.51	2.9017.4
T7 =	- - - - - - - - - - 707.95	1047.8
T8 =	- - - - - - - - - - 33.45	46.1
T9 =	- - - - - - - - - - - 1.83	2.3
	S =	.0953.2277.9839.9238.1

Adjiciatur jam S aggregato initialium, & obtinebis 1.3110.2877.7146.0598.7 pro valore Seriei, id est, pro longitudine Curvæ Elasticæ, modo in lineam rectam extensa foret. Hunc autem numerum determinavit *Bernoullius* consistere inter limites 1.308 & 1.315. Quod si longitudini Elasticæ adjiciatur sua Ordinata, habebitur numerus 1.9100.9889.4513.8559.8 qui est semiperiferia Ellipseos habentis 1 & $\sqrt{2}$ pro Axibus. Et hæc Exempla sufficiant; haud enim immoror Seriebus quæ per hanc Propositionem summari possint accurate.

Scholion.

Hoc Theorema nullo fere negotio exhibet Areas Curvarum binomialium quarum Ordinatæ continentur sub forma $x^\theta \times \overline{e+fx^\eta}^\lambda$, in unico tamen casu, scilicet ubi est $e+fx^\eta=0$, sive $x^\eta=-\frac{e}{f}$; hoc est in eo casu in quo Series pro Area convergit lentissime. Sed ubi Areæ non producendæ sunt ultra octo aut novem figuras, sufficit quærere Summam quatuor Terminorum initialium, nam S dabit eam reliquorum exiguo labore; imo si nulli Termini initiales colligantur, sed inchoetur transmutatio ad primum Terminum, valor ipsius S approximabit ad valorem totius Seriei sat celeriter. Series autem in Theoremate, redditur generalior, atque extenditur ad casus qui ad Quadraturas non attinent, ut sequitur. Sit Æquatio ad Seriem $T'=\frac{zz+m'}{zz+nz+r}T$, & ponatur

$$a = n - \frac{r-m}{n}, \qquad T2 = \frac{T}{n-1} \times \frac{ma + \overline{n-a} \times r}{zz + nz + r},$$

$$b = n + 2 - \frac{r-m+n+1}{n+2}, \qquad T3 = \frac{T2}{n+1} \times \frac{mb + \overline{n-b+2} \times \overline{r+n+1}}{zz + \overline{n+2}.z + r + n + 1},$$

$$c = n + 4 - \frac{r-m+2n+4}{n+4}, \qquad T4 = \frac{T3}{n+3} \times \frac{mc + \overline{n-c+4} \times \overline{r+2n+4}}{zz + \overline{n+4}.z + r + 2n + 4},$$

$$d = n + 6 - \frac{r-m+3n+9}{n+6}, \qquad T5 = \frac{T4}{n+5} \times \frac{md + \overline{n-d+6} \times \overline{r+3n+9}}{zz + \overline{n+6}.z + r + 3n + 9},$$

$$e = n + 8 - \frac{r-m+4n+16}{n+8}, \qquad T6 = \frac{T5}{n+7} \times \frac{me + \overline{n-e+8} \times \overline{r+4n+16}}{zz + \overline{n+8}.z + r + 4n + 16},$$

&c. &c.

Eritque S =

$$\frac{z+a-1}{n-1} T + \frac{z+b-1}{n+1} T2 + \frac{z+c-1}{n+3} T3 + \frac{z+d-1}{n+5} T4 + \frac{z+e-1}{n+7} T5 + \text{\&c.}$$

Exemplum.

Proponatur Series $\frac{1}{2} + \frac{1.1}{2.4} A + \frac{3.3}{4.6} B + \frac{5.5}{6.8} C + \frac{7.7}{8.10} D + \frac{9.9}{10.12} E$ &c. quæ definitur Æquatione $T' = \frac{z}{z+\frac{1}{2}} \times \frac{z}{z+\frac{3}{2}} T$, exiſtentibus $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. valoribus Abſciſſæ z ſucceſſivis. Hæc Æquatio comparari nequit cum $T' = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, eâ ſcilicet in Propoſitione; in qua utique ſunt duo factores $z-n$, & $z-n+1$ unitate differentes, quorum alter eſt in Numeratore, alter in Denominatore: quum tamen in Æquatione propoſitam Seriem definiente differentiæ factorum in Numeratore & Denominatore ſint $\frac{1}{2}$ & $\frac{3}{2}$. Igitur multiplico factores in ſe, & prodit $T' = \frac{zz}{zz+2z+\frac{3}{4}} T$, & pergo ad Æquationem in Scholio; quam cum altera comparans, habeo $m = 0$, $n = 2$, & $r = \frac{3}{4}$; & hiſce ſcriptis oriuntur

$a =$

$a = \frac{13}{8}$, $\quad T2 = \frac{1}{8} \times \frac{9}{4zz + 8z + 3} T$,

$b = \frac{49}{16}$, $\quad T3 = \frac{3}{16} \times \frac{25}{4zz + 16z + 15} T2$,

$c = \frac{109}{24}$, $\quad T4 = \frac{5}{24} \times \frac{49}{4zz + 24z + 35} T3$,

$d = \frac{193}{32}$, $\quad T5 = \frac{7}{32} \times \frac{81}{4zz + 32z + 63} T4$,

$e = \frac{301}{40}$, $\quad T6 = \frac{9}{40} \times \frac{121}{4zz + 40z + 99} T5$,

&c. &c.

Et $S = \frac{8z+5}{1.8} T + \frac{16z+33}{3.16} T2 + \frac{24z+85}{5.24} T3 + \frac{32z+161}{7.32} T4 +$ &c.

Quære jam aggregatum ſex Terminorum initialium, idemque prodibit .6106.6818.2373.0. Tum ſubſtitue ſeptimum Terminum pro T, & valorem ſeptimum $\frac{13}{2}$ pro z; & per calculum invenies

T =	.0036.3492.9656.98	.0258.9887.3806.0
T2 =	1825.5785.11	5210.5053.3
T3 =	29.7131.92	59.6739.9
T4 =	8425.62	1.3879.7
T5 =	339.30	491.0
T6 =	17.50	23.1
T7 =	1.09	1.3
		S = .0259.5158.9994.3

Inventa hoc modo S, eadem addatur Summæ initialium & prodibit .6366.1977.2367.3 pro valore Seriei propoſitæ. Si vero unitas dividatur per hunc numerum, quotus dabit Aream Circuli. Et eadem ratione tranſmutantur Series quæ definiuntur Æquatione hujuſmodi $T' = \frac{z^3 + az^2 + bz + c}{z^3 + dz^2 + ez + f} T$, vel etiam generaliori. Et notandum eſt Æquationem in Scholio $T' = \frac{zz + m}{zz + nz + r} T$, non ad pauciores caſus ideo extendi quod in Numeratore deſit membrum in quo z eſt unius dimenſionis: illud enim ſemper tolli poteſt mutando principium Abſciſſæ, eoque pacto Theorema redditur ſimplicius.

PROPO-

PROPOSITIO XII.

Oporteat transmutare Seriem definitam Æquatione hujus formæ

$T\times\overline{z^{\theta}+az^{\theta-1}+bz^{\theta-2}+\&c}+rT'\times\overline{z^{\theta}+cz^{\theta-1}+dz^{\theta-2}+\&c.}=0.$

in aliam celerrime convergentem.

Assume $m=c-\frac{b-d}{a-c}-\frac{1}{r+1}$, $n=m+\frac{a-c}{r+1}$; atque per Propositionem decimam erit $\frac{r}{r+1}\times\frac{z+m}{z+n}T$ proxime æqualis Seriei. Sic itaque quantitas illa primus Terminus Seriei transmutatæ: & ad inveniendum secundum, ponatur $S=\frac{r}{r+1}\times\frac{z+m}{z+n}T+S2$; deinde per Propositionem nonam quæratur Æquatio ad Terminos Seriei S2; & ex illa eliciatur quantitas proxime æqualis Seriei S2, eodem prorsus modo ac approximatio ad Seriem S prius inventa fuit; & quantitas illa erit secundus Terminus Seriei transmutatæ: ac progrediendo super hisce vestigiis, invenire licet Terminos sequentes tot quot volueris. Q. E. I.

EXEMPLUM I.

Transmutanda sit Series $\sqrt{12}$ in $1-\frac{1}{3.3}+\frac{1}{5.9}-\frac{1}{7.27}+\frac{1}{9.81}-\frac{1}{11.243}+$ &c. quam adhibuit *Halleius* in Quadratuta Circuli. Relatio Terminorum definitur Æquatione $zT+3T'\overline{z+1}=0$, in qua valores ipsius z sunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. Comparando autem hanc cum Æquatione in Problemate, erit $r=3$, $a=0$, $b=0$, $c=1$, $d=0$; & inde $m=\frac{3}{4}$, $n=\frac{1}{2}$; quibus scriptis habebitur $\frac{3}{4}\times\frac{4z+3}{4z+2}T$ pro primo Termino Seriei transmutatæ.

Finge $S=\frac{3}{4}\times\frac{4z+3}{4z+2}T+S2$, tum scribendo valores variabilium suc-

cedentes pro antecedentibus, proveniet $S - T = \frac{3}{4} \times \frac{4z+7}{4z+6} T' + S2 -$ $T2$, quarum Æquationum differentia dabit $T = \frac{3}{4} \times \frac{4z+3}{4z+2} T - \frac{3}{4} \times \frac{4z+7}{4z+6} T' + T2$, & exinde $T2 = \frac{1}{4} \times \frac{4z-1}{4z+2} T + \frac{3}{4} \times \frac{4z+7}{4z+6} T'$. Sed per Æquationem ad Seriem est $T' = -\frac{1}{3} T \times \frac{z}{z+1}$; quo substituto, obtinebis $T2 = -\frac{1}{8} \times \frac{1}{2z+1} \times \frac{2}{2z+2} \times \frac{3}{2z+3} T$. In quo valore scribendo quantitates variabiles posteriores pro prioribus, emerget $T'2 = -\frac{1}{8} \times \frac{1}{2z+3} \times \frac{2}{2z+4} \times \frac{3}{2z+5} T'$, vel substituendo $-\frac{1}{3} T \times \frac{z}{z+1}$ pro T', eveniet $T'2 = +\frac{1}{8} \times \frac{1}{2z+3} \times \frac{2}{2z+4} \times \frac{1}{2z+5} \times \frac{z}{z+1} T$; per hunc valorem ipsius $T'2$, est $T = \frac{2}{z} \times \overline{2z+2} \times \overline{2z+3} \times \overline{2z+4} \times \overline{2z+5}\, T'2$: Et per valorem ipsius $T2$ antea repertum est $T = -\frac{4}{3} \times \overline{2z+1} \times \overline{2z+2} \times \overline{2z+3}\, T2$. Denique æquando hosce duos valores, Termini T, prodibit $\overline{zz + \frac{1}{2} z} \times T2 + 3T'2 \times \overline{zz + \frac{9}{2} z + 5} = 0$, Æquatio utique ad Terminos Seriei $S2$; quæ collata cum illa in Propositione dat $r = 3$, $a = \frac{1}{2}$, $b = 0$, $c = \frac{9}{2}$, $d = 5$; hinc fit $m = 3$, $n = 2$; & proinde erit $\frac{3}{4} \times \frac{z+3}{z+2} T2$ proxime æqualis Seriei $S2$; adeoque Terminus secundus Seriei transmutatæ.

Ad inveniendum tertium assumatur $S2 = \frac{3}{4} \times \frac{z+3}{z+2} T2 + S3$, & inde per methodum Differentialem erit $S2 - T2 = \frac{3}{4} \times \frac{z+4}{z+3} T'2 + S3$ $- T3$, qua de priore subducta, manet $T2 = \frac{3}{4} \times \frac{z+3}{z+4} T2 - \frac{3}{4} \times \frac{z+4}{z+3} T'2$ $+ T3$, ex qua oritur $T3 = \frac{1}{4} \times \frac{z-1}{z+2} T2 + \frac{3}{4} \times \frac{z+4}{z+3} T'2$. Sed per

Æqua-

Æquationem ad Seriem S2, eſt $T'2 = -\frac{1}{3} \times \frac{z}{z+2} \times \frac{2z+1}{2z+5} T2$; quo ſubſtituto in valore ipſius T3, provenit $T3 = -\frac{1}{8} \times \frac{4}{2z+4} \times \frac{5}{2z+5} \times \frac{6}{2z+6} T2$: ubi ſcribendo valores variabilium conſequentes pro antecedentibus, orietur $T'3 = -\frac{1}{8} \times \frac{4}{2z+6} \times \frac{5}{2z+7} \times \frac{6}{2z+8} T'2$; vel ſubſtituendo pro T′2 ſuum valorem, erit $T'3 = \frac{1}{8} \times \frac{4}{2z+6} \times \frac{5}{2z+7} \times \frac{6}{2z+8} \times \frac{1}{3} \times \frac{z}{z+2} \times \frac{2z+1}{2z+5} T2$. Per valores Terminorum T3, & T′3, ſi eliminetur T2, reſultabit $\overline{zz + \frac{1}{2}z}\, T3 + 3\, T'3 \times \overline{zz + \frac{15}{2}z + 14} = 0$, quæ eſt Æquatio ad Terminos Seriei S3; eaque comparata cum illa in Theoremate exhibet $a = \frac{1}{2}$, $b = 0$, $c = \frac{15}{2}$, $d = 14$; adeoque $m = \frac{21}{4}$, $n = \frac{14}{4}$; & propterea $\frac{3}{4} \times \frac{4z+21}{4z+14} T3$ erit approximatio ad Seriem S3, ſive Terminus tertius Seriei tranſmutatæ.

Et ſimili proceſſu, ponendo $S3 = \frac{3}{4} \times \frac{4z+21}{4z+14} T3 + S4$, invenies Terminum quartum eſſe $\frac{3}{4} \times \frac{4z+30}{4z+20} T4$; exiſtente $T4 = -\frac{1}{8} \times \frac{7}{2z+7} \times \frac{8}{2z+8} \times \frac{9}{2z+9} T3$. Atque eadem omnino ratione Series tranſmutata ad libitum producetur. Sed jam Terminorum progreſſio apparet, exiſtentibus

$$T2 = -\frac{1}{8} \times \frac{1}{2z+1} \times \frac{2}{2z+2} \times \frac{3}{2z+3} T,$$

$$T3 = -\frac{1}{8} \times \frac{4}{2z+4} \times \frac{5}{2z+5} \times \frac{6}{2z+6} T2,$$

$$T4 = -\frac{1}{8} \times \frac{7}{2z+7} \times \frac{8}{2z+8} \times \frac{9}{2z+9} T3,$$

$$T5 = -\frac{1}{8} \times \frac{10}{2z+10} \times \frac{11}{2z+11} \times \frac{12}{2z+12} T4,$$

&c.

Ac

Ac $S = \frac{3}{4}$ in $\frac{4z+3}{4z+2}T + \frac{4z+12}{4z+8}T2 + \frac{4z+21}{4z+14}T3 + \frac{4z+30}{4z+20}T4 +$ &c.

Summa decem Terminorum initialium eſt 3.1415.9051.0938.0800.9964.2 ; & decimus primus ſubſtitutus pro T, ac $\frac{21}{2}$ pro z, dant

$+T =$	.0000.0279.3565.0014.1347.8
$-T2 =$	172.5274.8297.5
$+T3 =$	1474.5938.7
$-T4 =$	3.8136.0
$+T5 =$	192.2
$-T6 =$	1.5

.0000.0214.2791.3363.1147.5	.0000.0000.0139.7472.6106.4
1244.1885.8	3.3215.2
171.7	1.4
+ .0000.0214.2791.4607.3205.0	− .0000.0000.0139.7475.9323.0

Subduc jam Summam negativorum ab ea affirmativorum, & proveniet .0000.0214.2651.7131.3882.0, quæ addita aggregato initialium exhibet 3.1415.9265.3589.7932.3846.2 pro valore Seriei propoſitæ. Et hi ſex Termini Seriei tranſmutatæ tantundem efficiunt ac trigintaduo Seriei ſimplicis. Sed horum Theorematum uſus latius patet in Seriebus lentiſſime convergentibus, ad quarum utique valores pervenire non licet per meram additionem Terminorum.

Exemplum II.

Tranſmutanda ſit Series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} +$ &c. cujus Æquatio eſt $zzT + \overline{zz+2z+1}\, T' = 0$, exiſtentibus 1, 2, 3, 4, 5, &c. valoribus ipſius z ordine ſuccedentibus. Hic erit $a=0$, $b=0$, $c=2$, $d=1$, $r=1$; unde $m=1$, $n=0$, & propterea $\frac{1}{2} \times \frac{z+1}{z} T$ eſt primus Terminus Seriei tranſmutatæ.

Ut eruatur ſecundus, finge $S = \frac{1}{2} \times \frac{z+1}{z} T + S2$; & per methodum Differentialem prodibit $S - T = \frac{1}{2} \times \frac{z+2}{z+1} T' + S2 - T2$; quæ

deducta

deducta de priore relinquit $T = \frac{1}{2} \times \frac{z+1}{z} T - \frac{1}{2} \times \frac{z+2}{z+1} T' + T2$; ex quâ invenietur $T2 = \frac{1}{2} \times \frac{z-1}{z} T + \frac{1}{2} \times \frac{z+2}{z+1} T'$: ſed Æquatio ad Seriem S dat $T' = \frac{-zzT}{zz+2z+1}$, quo ſubſtituto, proveniet $T2 = -\frac{2z+1}{2z} \times \frac{T}{\overline{z+1}|^3}$. Scribe jam valores variabilium conſequentes pro antecedentibus, & orietur $T'2 = -\frac{2z+3}{2z+2} \times \frac{T}{\overline{z+2}|^3}$, vel rurſus ponendo pro T' ſuum valorem, $T'2 = \frac{1}{2} \times \frac{2z+3}{\overline{z+1}|^3} \times \frac{zz}{\overline{z+2}|^3} T$. Per valores Terminorum $T2$ & $T'2$, extermina T, & habebis $\overline{z^4 + \frac{3}{2} z^3} \times T2 + \overline{z^4 + \frac{13}{2} z^3 + 15z^2 + 14z + 4} \times T'2$, quæ eſt Æquatio ad Terminos Seriei $S2$: eaque collata cum illa in Propoſitione, dat $a = \frac{3}{2}$, $b = 0$, $c = \frac{13}{2}$, $d = 15$, $r = 1$; & proinde $m = 3$, $n = \frac{1}{2}$; quapropter eſt $\frac{z+3}{2z+1} T2$ valor Seriei $S2$ quamproxime, ſive Terminus ſecundus Seriei tranſmutatæ. Atque ſuper hiſce veſtigiis progrediendo ut in Exemplo ſuperiore, invenies

$$T2 = -\frac{2z+1}{2z} \times \frac{T}{\overline{z+1}|^3},\ T3 = -\frac{2z+2}{2z+1} \times \frac{8T2}{\overline{z+2}|^3},\ T4 = -\frac{2z+3}{2z+2} \times \frac{27T3}{\overline{z+3}|^3}, \&c.$$

$$\text{Ac } S = \frac{z+1}{2z} T + \frac{z+3}{2z+1} T2 + \frac{z+5}{2z+2} T3 + \frac{z+7}{2z+3} T4 + \&c.$$

Vel ob faciliorem computum, pone

$$A = T,\ B = \frac{A}{\overline{z+1}|^3},\ C = \frac{8B}{\overline{z+2}|^3},\ D = \frac{27C}{\overline{z+3}|^3},\ E = \frac{64D}{\overline{z+4}|^3}, \&c.$$

$$\text{Eritque } S = \frac{1}{2z} \text{ in } \overline{z+1}\,A - \overline{z+3}\,B + \overline{z+5}\,C - \overline{z+7}\,D + \overline{z+9}\,E - \&c.$$

Summa decem Terminorum Seriei ſummandæ ſub propriis ſignis eſt .8179.6217.5610.9851.3. Tum ut habeatur Summa reliquorum ſubſtitue Terminum decimum primum, hoc eſt, $\frac{1}{121}$ pro T, & pro z valorem ſuum reſpectivum, ſcilicet 11; ac habebis

$A = \frac{1}{121}$, $B = \frac{1.1.1}{12.12.12} A$, $C = \frac{2.2.2}{13.13.13} B$, $D = \frac{3.3.3}{14.14.14} C$, &c.

Et $S = \frac{1}{11}$ in $6A - 7B + 8C - 9D + 10E -$ &c.

Computus autem ſic ſe habet

A =	.0082.6446.2809.9173.5
B =	478.2675.2372.2
C =	1.7415.2944.5
D =	171.3604.0
E =	3.2495.0
F =	991.7
G =	43.6
H =	2.6
I =	.2

Deinde

.0495.8677.6859.5041.0	.0000.3347.8726.6605.4
13.9322.3556.0	1542.2436.0
32.4950.0	1.0908.7
523.2	33.8
2.8	— .0000.3348.0296.9983.9
+ .0495.8691.6214.4073.0	

Differentia harum Summarum diviſa per 11, dat S = .0045.0485.7813.1280.8, quæ adjecta aggregato decem initialium, conficit .8224.6703.3424.1132.1 pro valore Seriei. Quumque hic numerus dimidium ſit illius in Exemplo primo Propoſitionis decimæ reperti; concludendum eſt utrumque computum recte inſtitutum fuiſſe. Etenim Series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} +$ &c. ubi Termini ſunt alternatim negativi & affirmativi, dimidium eſt hujus $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ &c. in qua omnes ſunt ejuſdem ſigni.

Exemplum III.

Proponatur Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ &c. quæ definitur Æquatione $zT + \overline{z+1}\,T' = 0$, in qua valores Abſciſſæ ſunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$,

$\frac{5}{2}$, $\frac{7}{2}$, &c. Et ineundo computum juxta hanc Propositionem, devenietur ad regulam sequentem. Pone $A = T$, deinde

$$B = \frac{1}{2} A \times \frac{1}{2z+1} \times \frac{1}{\overline{z+1}|^2},$$

$$C = \frac{1}{2} B \times \frac{2}{2z+3} \times \frac{9}{\overline{z+2}|^2},$$

$$D = \frac{1}{2} C \times \frac{3}{2z+5} \times \frac{25}{\overline{z+3}|^2},$$

$$E = \frac{1}{2} D \times \frac{4}{2z+7} \times \frac{49}{\overline{z+4}|^2},$$

&c.

Eritque $S = \frac{1}{4z}$ in $\overline{2z+1}A - \overline{2z+5}B + \overline{2z+9}C - \overline{2z+13}D + \overline{2z+17}E -$ &c.

Summa duodecem Terminorum in Serie proposita est .7646.0069.1481.8329.5, atque Terminus decimus tertius $\frac{1}{25} = T$, & $z = \frac{25}{2}$, quibus scriptis, habemus $A = \frac{1}{25}$,

$B = \frac{1.1.1}{13.27.27} A$, $B = \frac{2.3.3}{14.29.29} B$, $D = \frac{3.5.5}{15.31.31} C$, $E = \frac{4.7.7}{16.33.33} D$, &c.

ac $S = \frac{1}{25}$ in $13\,A - 15\,B + 17\,C - 19\,D + 21\,E -$ &c.

A =	.0400.0000.0000.0000.0
B =	422.0744.9614.9
C =	6452.6422.0
D =	33.5725.4
E =	3776.5
F =	73.4
G =	2.2
H =	1

.5200.0000.0000.0000.0	.0000.6331.1174.4223.5
10.9694.9174.0	637.8782.6
7.9306.5	1688.2
55.0	2.7
+ .5200.0010.9702.5835.5	— .0000.6331.1812.4697.0

Dein

Dein dividendo differentiam Summarum per 25, prodit S = .0207.9747.1915.6153.5, quæ una cum aggregato initialium conficit .7853.9816.3397.4483.0 pro valore Seriei ſummandæ.

SCHOLION.

Sicut una Æquatio definit Series numero infinitas, ita una Tranſmutatio inſervit Seriebus numero infinitis: Et unumquodque Exemplum pro Theoremate eſt habendum; ſic Tranſmutatio in ultimo Exemplo inſervit Seriei generali

$$\frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \frac{1}{m+4n} - \&c.$$

Obſervandum eſt legem continuandi Series per hanc Propoſitionem tranſmutatas, non ſemper ſe prodere ut in Exemplis quæ hic elegi: hoc vero neutiquam incommodabit opus. Nam poſtquam colliguntur ſex circiter Termini Seriei ſummandæ, tres vel quatuor Seriei tranſmutatæ dabunt quæſitum ſatis accurate pro uſibus quibuſcunque; in praxi enim raro opus eſt continuare computum ultra novem aut decem figuras. Atque eodem res redit, ſive Termini Seriei tranſmutandæ ſint iiſdem vel contrariis ſignis affecti, ſive ſint vel non ſint aſſignabiles. Labor enim ſemper erit levis, præterquam ubi in Æquatione ad Seriem, quantitas r eſt negativa & ſimul proxime æqualis unitati; ut ſi ſit $r = -\frac{9}{10}$, vel $= -\frac{99}{100}$, computus erit laborioſus. Sed hoſce caſus facile evitabit Analyſta peritus, quibus igitur remedium proferre non eſt operæ pretium.

Adjungerem quoque nonnulla de Seriebus hujuſmodi

$$x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \&c.$$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \&c.$$

Ubi Termini in infinitum producti non habent datam rationem ad ſe invicem, ut in Seriebus de quibus hactenus egimus, ſed antecedentes ſunt infinite majores conſequentibus. Hæ exhibent Numerum ex dato Logarithmo, vel Sinum ex dato Arcu, & ſunt ſui generis ſimpliciſſimæ. Tranſmutari poſſunt per principia in ſuperioribus poſita; cæterum res commodius abſolvitur abſque Tranſmutationibus. Ut in Serie $x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \&c.$ quæ exhibet Numerum ex dato Logarithmo

rithmo x, si esset $x = 12.3785$, rejicerem Characteristicam 12, & quærerem Numerum Logarithmi .3785 qui prodiret in Serie celeriter convergente ob Logarithmum jam unitate minorem; quo dato non lateret Numerus Logarithmi 12.3785. Et eodem res redit sive Logarithmus sit Tabularis sive Hyperbolicus.

Et similiter in quærendo Sinum ex dato Arcu, si idem major sit Quadrante, subducatur ex Semicirculo, & manebit Arcus Quadrante minor eundem habens Sinum ac prior, utique ejus supplementum ad Semicirculum. Arcus autem minor Quadrante dabit suum Sinum in Serie celeriter convergente.

Series quæ definiuntur Æquationibus quæ involvunt tres vel plures Terminos Seriei possunt summari accurate vel quamproxime ex Analysi supra tradita: sufficiat autem hujusmodi computationum jecisse fundamentum & aperuisse viam aliis quibus est otium & animus hâc de materiâ plura scutari. Sed ne omnino neglecta videatur, dabimus Theorema generale ex principiis *Moivræi*, quod tam ad Summationem quam ad Transmutationem hujus generis Serierum extenditur.

PROPOSITIO XIII.

In Seriebus ex Divisione ortis, eadem est relatio inter Terminos ac Summas successivas.

Sit Fractio $\frac{1}{1-3x+xx}$, quæ in Seriem resoluta est $1+3x+8x^2+21x^3+55x^5+$ &c. Tum Summæ successivæ erunt

$$\frac{1}{1-3x+xx} = 1+3x+8x^2+21x^3+55x^4+144x^5+ \&c.$$

$$\frac{3x-xx}{1-3x+xx} = 3x+8x^2+21x^3+55x^4+144x^5+ \&c.$$

$$\frac{8x^2-3x^3}{1-3x+xx} = 8x^2+21x^3+55x^4+144x^5+ \&c.$$

$$\frac{21x^3-8x^4}{1-3x+xx} = 21x^3+55x^4+144x^5+ \&c.$$

$$\frac{55x^4-21x^5}{1-3x+xx} = 55x^4+144x^5+ \&c.$$

&c.

T Et

Et ſenſus Propoſitionis eſt quod hæ Summæ eodem numero ſumptæ eandem ubique habent relationem ac Termini Seriei totidem numero ſumpti. Exempli gratiâ, in caſu præſente relatio inter tres quoſvis Terminos ordine ſuccedentes erit $xxT - 3xT' + T'' = 0$: atque ea de cauſa relatio inter tres Summas ſucceſſivas erit etiam $xxS - 3xS' + S'' = 0$; ut experienti patebit. Propoſitio vero ſic demonſtratur.

Sint r, s, t datæ quantitates, & aſſumatur Æquatio ad Summas $rS + sS' + tS'' = 0$, dein ſubſtituendo valores variabilium ſuccedentes pro præſentibus, habebitur $rS' + sS'' + tS''' = 0$, quæ ſubducta de priori, relinquit $rS - rS' + sS' - sS'' + tS'' - tS''' = 0$; in hâc ſubſtitue T pro $S - S'$, T′ pro $S' - S''$, ac T″ pro $S'' - S'''$, & proveniet $rT + sT' + tT'' = 0$. Et hæc eſt eadem relatio ac ea Summarum primo aſſumpta. Et ſi plures vel pauciores ſint Summæ, eodem prorſus modo Propoſitio demonſtrabitur.

Corollarium. Hinc habemus methodum ſummandi haſce Series ex data relatione Terminorum; ut ſequentibus Exemplis fiet manifeſtum.

Exemplum I.

Detur Æquatio ad Terminos $rT + sT' = 0$, & per hanc Propoſitionem erit etiam eadem relatio Summarum, utique $rS + sS' = 0$; pro S′ ſubſtitue ſuum valorem $S - T$, & orietur $rS + sS - sT = 0$, unde fit $S = \frac{s}{r+s}T$. Quare datur Summa S ex dato ejus primo Termino T. Ut ſi Series ſit

$$1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \frac{16}{x^4} + \frac{32}{x^5} + \&c.$$

cujus Æquatio eſt $2T - xT' = 0$, erit $r = 2$, $s = -x$; quibus ſcriptis, prodibit $S = \frac{-x}{2-x}T$, vel $S = \frac{x}{x-2}T$. Subſtituatur jam quilibet Terminus pro T, & $\frac{x}{x-2}T$ erit Summa ejus & omnium ſequentium uſque in infinitum. Sit T æqualis primo Termino nempe unitati, & habebitur $\frac{x}{x-2}$ pro valore totius Seriei.

Exemplum II.

Ad eundem modum ſi Æquatio ad tres Terminos ſit $rT + sT' + tT'' = 0$, relatio Summarum erit $rS + sS' + tS'' = 0$; in quâ ſcribendo

do $S-T$ pro S', & $S-T-T'$ pro S'', emerget $rS+sS-sT+tS-tT-tT'=0$, & exinde $S=\frac{\overline{s+t}\,T+tT'}{r+s+t}$: adeoque datur S ex datis duobus Terminis. Sit Series summanda

$$1+3x+8x^2+21x^3+55x^4+144x^5+\&c.$$

in quâ relatio Terminorum est $xxT-3xT'+T''=0$, hinc est $r=xx$, $s=-3x$, $t=1$; quibus scriptis, habebitur $S=\frac{\overline{1-3x}\,T+T'}{1-3x+xx}$. Substitue jam primum Terminum pro T, & secundum pro T', & prodibit $\frac{1}{1-3x+xx}$ pro valore Seriei.

Similiter si Æquatio ad Terminos sit $rT+sT'+tT''+vT'''=0$, erit $S=\frac{\overline{s+t+v}\,T+\overline{t+v}\,T'+vT''}{r+s+t+v}$. Et sic deinceps quando relatio est inter plures Terminos.

Scholion.

Notandum relationem Terminorum quæ variabilis est, eo propius accedere ad invariabilem, quo magis Termini recedunt ab initio, & tandem in distantia infinita evadere omnino invariabilem ut in Seriebus ex Divisione ortis. Atque hanc appello *ultimam relationem* Terminorum ad quam relatio eorum continue approximat, nunquam tamen perveniet accurate priusquam Termini remoti fuerint infinito intervallo a principio.

Æquatio autem Differentialis definiens Seriem, rejiciendo dignitates Abscissæ omnes præter altissimam, & per hanc dividendo Æquationem residuam, dabit ultimam relationem Terminorum. Proponatur Æquatio $xxT\times\overline{zz+3z}=T'\times\overline{zz-5z+2}$, rejice omnes dignitates Abscissæ infra quadratum, & manebit $xxTzz=T'zz$, quæ divisa per zz, dabit $xxT=T'$. Et hæc est relatio ultima Terminorum.

Ultima autem relatio, quum constans sit, suppeditat methodum summandi Series quamproxime, in quibus relatio Terminorum est variabilis. Si detur Æquatio quævis $rT\times\overline{zz+az+b}+sT'\times\overline{zz+cz+d}=0$, ultima relatio Terminorum erit $rT+sT'=0$, unde $S=\frac{s}{r+s}T$ quamproxime. Hæc Æquatio obtinet accurate quando Terminus T infinite distat ab initio, & proxime quando distantia est notabiliter magna. Similiter si Æquatio sit $rT\times\overline{z+a}+sT'\times\overline{z+b}+tT''\times\overline{z+c}$, ultima

ultima relatio erit $rT + sT' + tT'' = 0$, & $S = \frac{\overline{s+t}T + tT'}{r+s+t}$ proxime. Adeoque colligendo aliquot Terminos initiales, priusquam inchoatur computus, Summa reliquorum habebitur proxime per hanc methodum. Ex iisdem principiis etiam corrigere licet approximationem continue ut in Propositione sequente.

Propositio XIV.

Omnis Series $A + B + C + D + E +$ *&c. in quâ ultima relatio Terminorum est* $rT + sT' + tT'' = 0$, *assumendo* $n = r + s + t$, *& ponendo*

$$\begin{array}{lll} A2 = rA + sB + tC, & A3 = rA2 + sB2 + tC2, & A4 = rA3 + sB3 + tC3, \\ B2 = rB + sC + tD, & B3 = rB2 + sC2 + tD2, & B4 = rB3 + sC3 + tD3, \\ C2 = rC + sD + tE, & C3 = rC2 + sD2 + tE2, & C4 = rC3 + sD3 + tE3, \text{ \&c.} \\ D2 = rD + sE + tF, & D3 = rD2 + sE2 + tF2, & D4 = rD3 + sE3 + tF3, \\ E2 = rE + sF + tG, & E3 = rE2 + sF2 + tG2, & E4 = rE3 + sF3 + tG3, \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}$$

bipartitur in sequentes

$$\overline{s+t} \text{ in } \frac{A}{n} + \frac{A^2}{n^2} + \frac{A^3}{n^3} + \frac{A^4}{n^4} + \frac{A^5}{n^5} + \text{\&c.}$$

$$+ t \text{ in } \frac{B}{n} + \frac{B2}{n^2} + \frac{B3}{n^3} + \frac{B4}{n^4} + \frac{B5}{n^5} + \text{\&c.}$$

Per hanc itaque Propositionem, Series in quâ ultima relatio Terminorum definitur Æquatione involvente tres Terminos, transmutabitur in duas hic exhibitas: quarum tamen prior evanescit ubi est $s + t = 0$. Si relatio sit inter duos tantum Terminos $rT + sT' = 0$, erit $t = 0$; eaque de causa, evanescet Series posterior, & ea quæ transmutanda proponitur, migrabit in unicam; quo in casu particulari coincidet hæc Propositio cum Theoremate a D. *Monmort* invento. Si relatio sit inter plures Terminos quam in hac Propositione, Series transmutanda migrabit in plures Series. At in omni casu res erit manifesta ex eo jam exhibito. Et nolim amittere simplicitatem, affectando nimium paucis complecti.

Exemplum

EXEMPLUM I.

Proponatur Series $1+4x+9x^2+16x^3+25x^4+36x^5+$ &c. Ubi coefficientes ſunt quadrata numerorum naturalium: Æquatio differentialis eſt $xT\times\overline{zz+2z+1}-T'zz=0$, & exinde ultima relatio $xT-T'=0$, adeoque $r=x$, $s=-1$, $t=0$, $n=x-1$; atque

$A=1$,	$A2=-3x$,	$A3=2x^2$,	$A4=0$,
$B=4x$,	$B2=-5x^2$,	$B3=2x^2$,	$B4=0$,
$C=9x^2$,	$C2=-7x^3$,	$C3=2x^2$,	&c.
$D=16x^3$,	$D2=-9x^4$,	&c.	
$E=25x^4$,	&c.		
&c.			

Omnes igitur Termini poſt $A3$, evaneſcunt; & ſubſtituendo hoſce valores, Series tranſmutata fit

$$-1 \text{ in } \frac{1}{x-1}-\frac{3}{\overline{x-1}^2}+\frac{2xx}{\overline{x-1}^3}.$$

Et hi tres Termini in unum collecti dant $\frac{1+x}{\overline{1-x}^3}$ pro valore Seriei. Quod ſi mutetur ſignum ipſius x tam in Serie, quam in ejus valore, habebitur

$$\frac{1-x}{\overline{1+x}^3}=1-4x+9x^2-16x^3+25x^4-36x^5+\&c.$$

Et inde etiam

$$\frac{1+6x^2+x^4}{\overline{1-xx}^3}=1+9x^2+25x^4+49x^6+81x^8+\&c.$$

id quod colligi poteſt directe ex Propoſitione abſque hiſce ambagibus.

EXEMPLUM II.

Proponatur Series $1+8x+27x^2+64x^3+125x^4+216x^5+$ &c. ubi Coefficientes ſunt cubi numerorum naturalium: & Æquatio ad Seriem eſt $xT\times\overline{z^3+3z^2+3z+1}-T'z^3=0$; adeoque ultima relatio $xT-T'=0$; & inde $r=x$, $s=-1$, $t=0$, $n=x-1$; &

$A=1$,	$A2=-7x$,	$A3=12x^2$,	$A4=-6x^3$,	$A5=0$,
$B=8x$,	$B2=-19x^2$,	$B3=18x^3$,	$B4=-6x^4$,	$B5=0$,
$C=27x^2$,	$C2=-37x^3$,	$C3=24x^4$,	$C4=-6x^5$,	&c.
$D=64x^3$,	$D2=-61x^4$,	$D3=30x^5$,	&c.	
$E=125x^4$,	$E2=-91x^5$,	&c.		
$F=216x^5$,	&c.			
&c.				

Hic evanefcunt Termini poft A4; & valores fubftituti in Theoremate, dant

$$-1 \text{ in } \frac{1}{x-1} - \frac{7x}{\overline{x-1}^2} + \frac{12x^2}{\overline{x-1}^3} - \frac{6x^3}{\overline{x-1}^4},$$

& hi quatuor Termini collecti in unum dant $\frac{1+4x+x^2}{\overline{1-x}^4}$ pro valore Seriei.

EXEMPLUM III.

Detur Series fummanda $1-6x+27x^2-104x^3+366x^4-1212x^5+3842x^6-11784x^7+$ &c. quæ definitur Æquatione $xxT\overline{z+4}-2xT'\overline{z+2}-T''z=0$, evadat jam z infinite magna, & ultima relatio erit $xxT-2xT'-T''=0$: hinc fit $r=xx$, $s=-2x$, $t=-1$, $n=xx-2x-1$; atque

$A=+1$,	$A2=-14x^2$,	$A3=+29x^4$,	$A4=0$,
$B=-6x$,	$B2=+44x^3$,	$B3=-70x^5$,	$B4=0$,
$C=+27x^2$,	$C2=-131x^4$,	$C3=+169x^6$,	&c.
$D=-104x^4$,	$D2=+376x^5$,	$D3=-408x^7$,	
$E=+366x^4$,	$E2=-1052x^6$,	&c.	
$F=-1212x^5$,	$F2=+2888x^7$,		
$G=+3842x^6$,	&c.		
$H=-11784x^7$			
&c.			

Subftitue jam hofce valores pro A, A2, A3, B, B2, B3, atque Series propofita tranfmigrabit in fequentes duas numero Terminorum finitas,

$$-\overline{2x+1} \text{ in } \frac{1}{xx-2x-1} - \frac{14x^2}{\overline{xx-2x-1}^2} + \frac{29x^4}{\overline{xx-2x-1}^3},$$
$$\frac{-6x}{xx-2x-1} + \frac{44x^3}{\overline{xx-2x-1}^2} - \frac{70x^5}{\overline{xx-2x-1}^3},$$

Quæ duæ in unam Summam collectæ dant $\frac{1}{\overline{1+2x-xx}^3}$ pro valore Seriei. Ad eundem modum Theorema applicatur ad Tranfmutationem Serierum quæ fummari nequeunt. Demonftratio autem ex fequentibus fiet

fiet manifesta. Sit Fractio quæcunque $\frac{a+bx+cx^2+dx^3}{\overline{v+tx+sx^2+rx^3}|^n}$, cujus Numerator sit quantitas constans ex membris quotcunque, quorum tamen numerus est determinatus. Sit vero ejus Denominator dignitas quælibet quantitatis, quæ etiam conflatur ex membris quotvis Numero finitis. Tum quicunque sit index n, si Fractio resolvatur in Seriem, eadem semper erit ultima relatio Terminorum, ac si Denominator fuisset simplex potestas $v+tx+sx^2+rx^3$. In hanc igitur si Series continue multiplicetur, factum tandem abrumpet, si sit n integer & affirmativus: hoc est, si Series summabilis sit per simplicem Æquationem.

Proponatur Series $1-6x+27x^2-104x^3+366x^4-$ &c. in quâ ultima relatio Coefficientium est $A-2B-C=0$; sumo quantitatem $xx-2x-1$, vel mutatis signis, $1+2x-xx$, in qua Coefficientes iidem sunt ac in ultima relatione, & concludo Seriem propositam, modo summabilis sit, æqualem esse alicui Fractioni cujus Denominator est Dignitas quædam quantitatis $1+2x-xx$. Ergo pono $S=1-6x+27x^2-104x^3+366x^4-$ &c. & duco utramque partem in $1+2x-xx$; atque prodit $S\times\overline{1+2x-xx}=1-4x+14x^2-44x^3+131x^4-$ &c. quam quoniam nondum abrumpit, iterum duco in eandem quantitatem, & habeo $S\times\overline{1+2x-xx}|^2=1-2x+5x^2-12x^3+29x^4-$ &c. dein tertia vice multiplicans eandem, obtineo $S\times\overline{1+2x-xx}|^3=1$; Terminis omnibus post primum evanescentibus.

Igitur palam est hanc Propositionem nihil aliud præstare quam methodum compendiosam multiplicationis, & simul dispositionem Terminorum in Seriebus transmutatis quæ efficit celerem convergentiam quando non abrumpunt. Quare Demonstrationem Lectori indagandam relinquo, quæ quidem eadem fere est sive Denominatores sint trinomiales, quadrinomiales aut plurium nominum.

PROPOSITIO XV.

Invenire Æquationem sive Algebraica sive Fluxionalis sit, cujus Radix erit Series quæcunque data quæ definitur Æquatione in qua Termini Seriei sunt unius tantum dimensionis.

Series datur ex datis aliquot Terminis initialibus & simul lege formandi reliquos: ex hisce autem datis Invenietur Æquatio habens Seriem illam pro Radice, id quod per Exempla sequentia intelligetur.

EXEMPLUM

EXEMPLUM I.

Invenire Æquationem cujus Radix eſt Series $A+Bx+Cx^2+Dx^3+Ex^4+$ &c. in quâ relatio Coefficientium eſt conſtans, ſcilicet $rA+sB+tC+vD=0$. Quoniam relatio eſt invariabilis, concludo Seriem æquari Fractioni rationali, quam ſic eruo. Fingo y æqualem Serie datæ: & ſumendo indices relationis r, s, t, v, in ordine inverſo, duco y ejuſque valorem ſucceſſive in v, tx, sx^2, rx^3, atque provenient

$$\begin{aligned} vy &= vA+vBx+vCx^2+vDx^3+vEx^4+vFx^5+\text{&c.} \\ txy &= \quad tAx+tBx^2+tCx^3+tDx^4+tEx^5+\text{&c.} \\ sx^2y &= \quad\quad sAx^2+sBx^3+sCx^4+sDx^5+\text{&c.} \\ rx^3y &= \quad\quad\quad rAx^3+rBx^4+rCx^5+\text{&c.} \end{aligned}$$

Sed ex ſuppoſitione, eſt $rA+sB+tC+vD=0$, $rB+sC+tD+vE=0$, & ſic in infinitum. Unde evaneſcunt omnes Termini in quibus x eſt plurium quam duarum dimenſionum; adeoque reſtabit

$$vy+txy+sx^2y+rx^3y=vA\left.\begin{matrix}+vB\\+tA\end{matrix}\right\}x+\left.\begin{matrix}+vC\\+tB\\+rA\end{matrix}\right\}x^2.$$

Quæ eſt Æquatio determinans valorem Seriei. Exempli cauſâ, ſit $A=2$, $B=-3$, $C=7$, $r=3$, $s=-7$, $t=6$, $v=4$; & evadet Æquatio

$$4y+6xy-7x^2y+3x^3y=8x-4x^3, \text{ vel } y=\frac{8x-4x^3}{4+6x-7x^2+3x^3}.$$

EXEMPLUM II.

Invenire Æquationem quæ pro Radice habeat Seriem $1-\frac{2}{3}x^2+\frac{8}{15}x^4-\frac{16}{35}x^6+\frac{128}{315}x^8-\frac{256}{693}x^{10}+$ &c. in qua relationes Coefficientium ſunt $2A+3B=0$, $4B+5C=0$, $6C+7D=0$, $8D+9E=0$, &c. In hoc Exemplo Æquatio quæſita involvet Fluxiones primas, quoniam numeri in relationibus ſunt æqui differentes. Aſſume ergo

$$\begin{aligned} y &= A+Bx^2+Cx^4+Dx^6+Ex^8+\text{&c.} \\ \text{erit } x\dot{y} &= \quad 2Bx^2+4Cx^4+6Dx^6+8Ex^8+\text{&c.} \\ y+x\dot{y} &= A+3Bx^2+5Cx^4+7Dx^6+9Ex^8+\text{&c.} \\ 2x^2y+x^3\dot{y} &= 2Ax^2+4Bx^4+6Cx^6+8Dx^8+\text{&c.} \end{aligned}$$

Adjiciatur nunc ultima Æquatio penultimæ atque prodibit

$y+$

$$\dot{y} + x\ddot{y} + 2x^2\dot{y} + x^3\ddot{y} = A \left.\begin{matrix} +2A \\ +3B \end{matrix}\right\} x^2 \left.\begin{matrix} +4B \\ +5C \end{matrix}\right\} x^4 \left.\begin{matrix} +6C \\ +7D \end{matrix}\right\} x^6 \left.\begin{matrix} +8D \\ +9E \end{matrix}\right\} x^8 + \&c.$$

Sed ex hypothesi relationes Coefficientium sunt $2A+3B=0$, $4B+5C=0$, $6C+7D=0$, &c. ideoque Termini omnes post primum A evanescunt, & restat Æquatio finita $\dot{y}+x\ddot{y}+2x^2\dot{y}+x^3\ddot{y}=A=1$, sive $\dot{y}\times\overline{1+2xx}+x\ddot{y}\times\overline{1+xx}=1$: substituta scilicet unitate, sive primo Termino Seriei pro Coefficiente A.

EXEMPLUM III.

Invenienda sit Æquatio cujus Radix est Series $1-\frac{1}{4}x^2-\frac{3}{64}x^4-\frac{5}{256}x^6-\frac{175}{2048}x^8$ — &c. ubi relationes Coefficientium sunt

$-1.1A-2.2B=0$, $1.3B-4.4C=0$, $3.5C-6.6D=0$, $5.7D-8.8E=0$, &c.

Vel $-A-4B=0$, $3B-16C=0$, $15C-36D=0$, $35D-64E=0$, &c.

In hoc casu Æquatio desiderata necessario involvet Fluxiones secundas, quoniam numeri qui indicant relationem Coefficientium, sunt duarum dimensionum, sive facta sub numeris æquidifferentibus binatim sumptis. Sit itaque

$$y=A+Bx^2+Cx^4+Dx^6+Ex^8+\&c.$$
$$\dot{y}=2Bx+4Cx^3+6Dx^5+8Ex^7+\&c.$$
$$x\ddot{y}=2Bx+12Cx^3+30Dx^5+56Ex^7+\&c.$$
$$\dot{y}+x\ddot{y}=4Bx+16Cx^3+36Dx^5+64Ex^7+\&c.$$
$$x^3\ddot{y}+x^2\dot{y}-xy=-Ax+3Bx^3+15Cx^5+35Dx^7+\&c.$$

Denique subducendo hanc ultimam Æquationem de penultima, manebit

$$x\ddot{y}-x^3\ddot{y}+\dot{y}-x^2\dot{y}+xy=\left\{\begin{matrix} +A \\ +4B \end{matrix}\right\} x \left.\begin{matrix} -3B \\ +16C \end{matrix}\right\} x^3 \left.\begin{matrix} -15C \\ +36D \end{matrix}\right\} x^5 \left.\begin{matrix} -35D \\ +64E \end{matrix}\right\} x^7 \&c.$$

hoc est $x\ddot{y}-x^3\ddot{y}+\dot{y}-x^2\dot{y}+xy=0$, vel $\dot{y}+x\ddot{y}+\frac{xy}{1-xx}=0$. Nam propter relationem Coefficientium, omnia membra ex una parte Æquationis evanescunt.

EXEMPLUM IV.

Quæratur Æquatio, cujus Radix est $1+\frac{1}{4}Ax^2+\frac{9}{16}Bx^4+\frac{25}{36}Cx^6$

$+\frac{49}{64}Dx^8+\&c.$ ubi relationes Coefficientium sunt $A-4B=0$, $9B-16C=0$, $25C-36D=0$, &c. Finge

$$y=A+Bx^2+Cx^4+Dx^6+Ex^8+Fx^{10}+\&c.$$

$$\text{Erit } \dot{y}= 2Bx+4Cx^3+6Dx^5+8Ex^7+10Fx^9+\&c.$$

$$\text{ac } x\ddot{y}= 2Bx+12Cx^3+30Dx^5+56Ex^7+90Fx^9+\&c.$$

Dein per computum invenies

$$xy+3x^2\dot{y}+x^3\dddot{y}=Ax+9Bx^3+25Cx^5+49Dx^7+81Ex^9+\&c.$$

$$\dot{y}+x\ddot{y}=4Bx+16Cx^3+36Dx^5+64Ex^7+100Fx^9+\&c.$$

Harum Æquationum posteriorem subducito de priore & habebis

$$xy-\dot{y}\times\overline{1-3xx}-x\ddot{y}\times\overline{1-xx}=\left\{\begin{matrix}+A\\-4B\end{matrix}\right\}x+\left\{\begin{matrix}+9B\\-16C\end{matrix}\right\}x^3+\left\{\begin{matrix}+25C\\-36D\end{matrix}\right\}x^5+\left\{\begin{matrix}+49D\\-64E\end{matrix}\right\}x^7 +\&c.$$

Et inde propter relationem Coefficientium, Æquatio erit $xy=\dot{y}\times\overline{1-3xx}+x\ddot{y}\times\overline{1-xx}$.

Exemplum V.

Quæratur Æquatio cujus Radix est $x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\&c.$ ubi Denominatores sunt quadrata numerorum 1, 2, 3, &c. Assume

$$y=x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+\&c.$$

$$\text{Erit } x\dot{y}=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\&c.$$

Cujus Fluxio est

$$x\ddot{y}+\dot{y}=1+x+x^2+x^3+x^4+\&c.$$

$$\text{hoc est } x\ddot{y}+\dot{y}=\frac{1}{1-x}.$$

Similiter si Series sit $x+\frac{1}{8}x^2+\frac{1}{27}x^3+\frac{1}{64}x^4+\&c.$ ubi Denominatores sunt Cubi, invenies Æquationem esse $\dot{y}+3x\ddot{y}+x^2\dddot{y}=\frac{1}{1-x}$.

Et si Series sit $x+\frac{1}{16}x^2+\frac{1}{81}x^3+\frac{1}{256}x^4\&c.$ existentibus Denominatoribus

toribus biquadratis, prodibit Æquatio $\dot{y} + 6x\ddot{y} + 7x^2\dddot{y} + x^3\ddddot{y} = \frac{1}{1-x}$. Et sic deinceps.

SCHOLION.

Ex data igitur relatione Coefficientium, Series reducuntur vel ad Fractiones vel ad Fluxiones, idque eâdem prorsus facilitate in omnibus casibus; sive Series determinetur per relationem plurium vel pauciorum Terminorum. Nam quoniam sumendo Fluxionem Seriei, Termini ducuntur in indices dignitatum, qui semper sunt æquidifferentes; & rursus in alias quantitates æquidifferentes, sumendo Fluxiones secundas; & sic porro, formare licet hac ratiône relationes Terminorum quæcunque sint, insistendo vestigiis mox traditis.

Ad summationem autem perfectam oportet Fluxiones reduci ad Fluentes; quod ubi fieri nequit, concludendum est, Seriem de qua agitur non esse ex ipsarum numero quæ summari possunt. Multum autem confert ad promovendam hanc doctrinam ut Summationes reducantur ad Fluxiones, quoniam methodus regrediendi a fluxionibus ad Fluentes, licet imperfecta sit, est tamen melius nota & magis exculta quam ea regrediendi a Differentiis ad Summas. Sed & altera alteri mutuo præbebit opem: Fluentes enim si inveniri possint, exhibebunt Summas; vel e contra Summæ inventæ dabunt Fluentes.

Ex Propositionibus jam traditis, aliisque ex iisdem principiis facile deducendis, invenire licet Radicem Æquationis cujuscunque accuratissime in numeris modo reduci possit in Seriem simplicem, etiamsi lentissime convergentem. Sed etiamnum restat difficultas, si neque Series cito approximet neque sit relatio Terminorum simplex & idonea quæ definiatur Æquatione differentiali. Quippe si in Æquatione resolvenda reperiatur Radix ejusve Fluxiones plurium quam unius dimensionis, vel si hæ in se mutuo multiplicentur; & hujusmodi membra nequeant eliminari: in iis inquam casibus Radix non est reducibilis in Seriem simplicem compositam ex dignitatibus Abscissæ, saltem per artes mihi hactenus notas. At dignitates Abscissæ utcunque altæ sint in Æquatione resolvendâ, nullatenus obstant simplicitati Seriei.

Sed hac oblata occasione, conabor tollere difficultatem hactenus intactam, quæ tamen in reductione Æquationum Fluxionalium maxima est. Sciendum igitur est eas Æquationes Fluxionales quæ involvunt omnes conditiones Problematis, ideoque determinant omnes Coefficientes in suis Radicibus, resolvi posse perinde ac Æquationes literales affectæ, idque per regulas a *Newtono* dudum traditas. Attamen plerumque fit Æquationes non determinare omnes Coefficientes Terminorum

rum in ſuis Radicibus; ſcilicet quoniam quantitates conſtantes evaneſcunt & omnino amittuntur ex Æquatione in progreſſu a Fluentibus ad Fluxiones. Sed & variis modis evaneſcere ſolent prout Fluxio ſumitur, id quod Exemplo declarare libet.

Proponatur Æquatio $y^2 = a^2 + bx + \frac{x^4}{c^2}$; & fluente x uniformiter, atque ſcripta unitate pro ejus Fluxione; habebitur per Methodum directam $2y\dot{y} = b + \frac{4x^3}{c^2}$: cujus Radix per methodos *Newtoni* extracta, eadem erit ac Radix Fluentis $y^2 = bx + \frac{x^4}{c^2}$, idque propter quantitatem aa jam amiſſam. Et hoc in caſu non determinatur primus Terminus Seriei, qui utique pendet ex evaneſcente quantitate aa.

Similiter ſi Æquatio propoſita prius dividatur per x, exibit $\frac{y^2}{x} = \frac{aa}{x} + b + \frac{x^3}{c^2}$, cujus Fluxio eſt $\frac{2y\dot{y}}{x} - \frac{y^2}{x^2} = \frac{-aa}{x^2} + \frac{3x^2}{cc}$, vel ducendo in x^2, $2xy\dot{y} - y^2 = -aa + \frac{3x^4}{cc}$. Et hujus Radix pure ſumpta eadem eſt ac Fluentis, $y^2 = a^2 + \frac{x^4}{cc}$. Quo in caſu non determinatur ſecundus Terminus Seriei, qui dependet ex Coefficiente b nunc amiſſâ.

Denique ſi Æquatio fluens dividatur per x^4, habebitur $\frac{y^2}{x^4} = \frac{aa}{x^4} + \frac{b}{x^3} + \frac{1}{cc}$, & exinde Fluxio erit $\frac{2y\dot{y}}{x^4} - \frac{4y^2}{x^5} = \frac{-4aa}{x^5} - \frac{3b}{x^4}$, vel $2xy\dot{y} - 4y^2 = -4aa - 3bx$; quæ reſoluta per regulas vulgares, eandem dabit Radicem atque Fluens $y^2 = a^2 + bx$: adeoque Terminus Seriei quintus, qui dependet ex Coefficiente cc non determinatur. Hinc patet fluxionales Æquationes

$$2y\dot{y} = b + \frac{4x^3}{c^2},$$

$$2xy\dot{y} - y^2 = -aa + \frac{3x^4}{cc},$$

$$2xy\dot{y} - 4y^2 = -4aa - 3bx$$

provenire ab unicâ fluente $y^2 = aa + bx + \frac{x^4}{cc}$: per primam Terminus primus Seriei non determinatur; per ſecundam determinatur primus at

non

non secundus Terminus; per tertiam denique determinantur primi quatuor Termini Seriei, at quintus relinquitur indeterminatus. Possunt igitur diversæ Æquationes fluxionales provenire ex eâdem fluente. Sed methodus resolutionis non erit perfecta priusquam in potestate fuerit exhibere omnes Radices diversarum fluentium a quibus Fluxio quævis proposita ullo modo pervenire possit. Nam necesse est habere rationem quantitatum quæ vel actu evanuere vel evanescere potuerunt.

Etenim non licet extrahere Radicem ex Æquatione fluxionali tanquam nulla quantitas evanuisset, dein addere quantitatem datam fluenti inventæ ut in Quadratura Curvarum. Termini utique indeterminati sæpissime sunt quantitates invariabiles: & additio Quantitatis datæ Radici inventæ non æquipollet additioni Quantitatis Æquationi resolvendæ. Sic eadem Fluxio $2y\dot{y}=b+\frac{4x^3}{cc}$ provenire potest ab utrâvis fluentium $yy=aa+bx+\frac{x^4}{cc}$, vel $yy=bx+\frac{x^4}{cc}$; quarum tamen Radices sortiuntur formas omnino diversas, scilicet prior $A+Bx+Cx^2+$ &c. posterior $Ax^{\frac{1}{2}}+Bx^{\frac{3}{2}}+Cx^{\frac{5}{2}}+$ &c.

Insuper fieri potest Æquationem fluxionalem involvere omnes Coefficientes quas involvit fluens, & hoc non obstante, Terminos omnes non determinari, ut in Exemplo sequente. Æquatio $y^2=aa+bx+\frac{x^4}{cc}$ exhibet directe Fluxionem $2y\dot{y}=b+\frac{4x^3}{cc}$: quæ si ducatur in y, evadet $2y^2\dot{y}=by+\frac{4x^3y}{cc}$; & dein substituatur pro yy suus valor $aa+bx+\frac{x^4}{cc}$, tandem habebitur $2aa\dot{y}+2bx\dot{y}+\frac{2x^4\dot{y}}{cc}=by+\frac{4x^3y}{cc}$. Et hæc est Æquatio fluxionalis involvens omnes Coefficientes quas involvit ejus fluens; in qua nihilominus primus Terminus Seriei non determinatur. Idem etiam ubique eveniet nisi adsit membrum in Æquatione quod nec involvit Radicem neque ejus Fluxionem quamlibet.

Sunt & aliæ difficultates quamplurimæ haud minoris momenti, quas facile quivis percipiet multiplicari in Fluxionibus inferiorum ordinum: nam in Fluxionibus secundis, Termini duo ab invicem neutiquam pendentes possunt esse indeterminati, & tres in tertiis; & sic porro. Cæterum accidit haud raro omnes Terminos determinari ex Æqaatione etiam involvente Fluxiones inferiorum Ordinum. Et omnia quæ hic dicta sunt de Fluxione cujus fluens nota est, etiam vera sunt de Fluxionibus quarum fluentes sunt ignotæ.

Radix Æquationis cujuscunque est quantitas, quæ scripta pro literâ denotante Radicem, efficiet omnes Terminos Æquationis resultantis evanescere. Termini autem duobus tantum modis possunt evanescere, vel per signorum contrarietatem in membris homologis, vel etiam ubi reperitur quantitas constans in fluente; eadem enim nullum relinquit vestigium in Fluxione. Ut si sit $y = Ax^n$; erit $\dot{y} = nAx^{n-1}$, $\ddot{y} = \overline{nn-n}Ax^{n-2}$; si sit $n=0$, evanescet valor Fluxionis primæ; si sit $nn-n=0$, id est, $n=0$, vel $n=1$, evanescet valor secundæ; idque absque aliis membris homologis quæ eas destruere possint. Et hæc sunt principia per quæ enodandæ sunt difficultates universæ quæ occurrunt in resolutione Æquationum fluxionalium.

Proponatur Æquatio resolvenda $r^2\dot{y}^2 = r^2\dot{x}^2 - \dot{x}^2y^2$, vel posita unitate pro $\dot{x}$, $r^2\dot{y}^2 = r^2 - y^2$. Per parallelogrammum vel alias *Newtoni* methodos, invenies Radicem

$$y = x - \frac{x^3}{6r^2} + \frac{x^5}{120r^4} - \frac{x^7}{5040r^6} + \&c.$$

Hujus enim quadratum scriptum pro y^2, & Fluxionis quadratum pro $\dot{y}^2$ efficiet omnia membra Æquationis resultantis sese destruere per contrarietatem signorum. Videamus autem an alia sit radix quæ hoc modo inveniri nequeat. Illum in finem finge Ax^n esse primum Terminum Seriei, sive $y = Ax^n$ proxime; eritque $\dot{y} = nAx^{n-1}$; adeoque $\dot{y}^2 = n^2A^2x^{2n-2}$, & $y^2 = A^2x^{2n}$; quibus scriptis, resultat $n^2r^2A^2x^{2n-2} = r^2 - A^2x^{2n}$, vel ducendo totam in x^2.

$$n^2r^2A^2x^{2n} = r^2x^2 - A^2x^{2n+2}.$$

Ubi patet membrum $n^2r^2A^2x^{2n}$ evanescere quando est $n=0$: substituatur ergo o pro n, & evadet Æquatio

$$0 \times r^2A^2x^0 = r^2x^2 - A^2x^2$$

hoc igitur in casu evanescit membrum $r^2A^2x^0$ in quo x est minimarum dimensionum: eâque de causâ erit Ax^0, sive constans quantitas A primus Terminus Seriei eo citius convergentis quo minor est x. Atque hunc computum prosequendo invenies per methodos vulgares

$$y = A \times \overline{1 - \frac{x^2}{2r^2} + \frac{x^4}{24r^4} - \frac{x^6}{720r^6} + \&c.}$$

Sed quantitas A ex Æquatione Fluxionali non determinatur. Quod si Æquationis $r^2\dot{y}^2 = r^2 - y^2$, capiatur Fluxio habebitur $2r^2\dot{y}\ddot{y} = -2y\dot{y}$, vel dividendo & transponendo $r^2\ddot{y} + y = 0$. Ponamus jam esse $y = Ax^n$, erit

erit $\dot{y}=nAx^{n-1}$, & $\ddot{y}=\overline{nn-n}\,Ax^{n-2}$, quibus ſubſtitutis, prodit $\overline{nn-n}\times r^2Ax^{n-2}+Ax^n=0$; ubi patet indicem $n-2$ non comparari poſſe cum indice n; & proinde nullam Radicem illâ methodo erui poſſe. Attamen ponendo Coefficientem $nn-n=0$, habebitur $n=0$, & $n=1$; adeoque poteſt A vel Ax eſſe primus Terminus Seriei, uti prius invenimus.

Prior Series dat Sinum, & poſterior Coſinum ex dato Arcu x. Eſtque Coefficiens A in poſteriore æqualis Radio r. Etenim ſive ponas y pro Sinu vel Coſinu, ſemper incides in Æquationem $r^2\dot{y}^2=r^2-y^2$, quæ nullâ Radice explicabilis eſt præter duas jam exhibitas. Ex hoc autem Exemplo notare licet formam Seriei ſemper determinatam eſſe ex Æquatione, etiamſi Coefficientes non determinentur. Et ubi Coefficiens cujuſvis Termini non determinatur, index ipſius x in eodem ſemper invenietur ponendo membrum aliquod Æquationis reſultantis æquale nihilo. Ubi vero Coefficiens Termini determinatur, index ipſius x in eodem invenitur per comparationem duorum indicum in Æquatione reſultante, idque per regulas *Newtoni*.

Reſolvenda ſit Æquatio $\ddot{y}+a^2y-x\dot{y}-x^2\ddot{y}=0$, ubi x fluit uniformiter, & unitas ſcribitur pro ejus Fluxione. Finge $y=Ax^n$ proxime, eritque $\dot{y}=nAx^{n-1}$, $\ddot{y}=\overline{nn-n}\,Ax^{n-2}$: quibus ſubſtitutis in Æquatione reſultat $\overline{nn-n}\,Ax^{n-2}+\overline{aa-nn}\,Ax^n=0$. Pone jam Coefficientem $nn-n=0$, & prodibit $n=0$, vel $n=1$: hi valores ſubſtituti pro n in Æquatione, exhibent

$$0\times Ax^{-2}+aaAx^0=0, \text{ \& }$$
$$0\times Ax^{-1}+\overline{aa-1}\,Ax=0;$$

quumque in utroque caſu evaneſcat membrum in quo x eſt minimarum dimenſionum, concludo eſſe 0 vel 1, indicem ipſius x in primo Termino Seriei quæ convergit eo citius quo minor eſt x: adeoque uſurpare licet

$$y=A+Bx^2+Cx^4+Dx^6+\text{\&c.}$$
$$\text{vel } y=Ax+Bx^3+Cx^5+Dx^7+\text{\&c.}$$

In Æquatione reſultante $\overline{nn-n}\,Ax^{n-2}+\overline{aa-nn}\,Ax^n=0$, pone nunc Coefficientem $aa-nn=0$; eritque $n=\pm a$: ſcribe igitur $+a$, & $-a$ pro n, & prodibit

$$\overline{aa-a}\,Ax^{a-2}+0\times Ax^a=0$$
$$\overline{aa+a}\,Ax^{-a-2}+0\times Ax^{-a}=0.$$

In utroque igitur caſu evaneſcit membrum in quo x eſt maximarum dimenſionum; eaque propter eritque $+a$ vel $-a$ index primi Termini Seriei eo citius convergentis quo major eſt x. Nam aſſumere licet

$y=$

$$y = Ax^{a} + Bx^{a-2} + Cx^{a-4} + Dx^{a-6} + \&c.$$

$$\text{vel } y = Ax^{-a} + Bx^{-a-2} + Cx^{-a-4} + Dx^{-a-6} + \&c.$$

Harum quatuor Serierum coefficientes determinatæ dant valores Radicis ſequentes

$$A + \frac{0-aa}{1.2}Ax^{2} + \frac{4-aa}{3.4}Bx^{2} + \frac{16-aa}{5.6}Cx^{2} + \frac{36-aa}{7.8}Dx^{2} + \&c.$$

$$Ax + \frac{1-aa}{2.3}Ax^{2} + \frac{9-aa}{4.5}Bx^{2} + \frac{25-aa}{6.7}Cx^{2} + \frac{49-aa}{8.9}Dx^{2} + \&c.$$

$$Ax^{a} - \frac{a.\overline{a-1}}{4.\overline{a-1}} \times \frac{A}{x^{2}} - \frac{\overline{a-2}.\overline{a-3}}{8.\overline{a-2}} \times \frac{B}{x^{2}} - \frac{\overline{a-4}.\overline{a-5}}{12.\overline{a-3}} \times \frac{C}{x^{2}} - \frac{\overline{a-6}.\overline{a-7}}{16.\overline{a-4}} \times \frac{D}{x^{2}} - \&c.$$

$$\frac{A}{x^{a}} - \frac{a.\overline{a+1}}{4.\overline{a+1}} \times \frac{A}{x^{2}} - \frac{\overline{a+2}.\overline{a+3}}{8.\overline{a+2}} \times \frac{B}{x^{2}} - \frac{\overline{a+4}.\overline{a+5}}{12.\overline{a+3}} \times \frac{C}{x^{2}} - \frac{\overline{a+6}.\overline{a+7}}{16.\overline{a+4}} \times \frac{D}{x^{2}} - \&c.$$

Duæ priores attinent ad multiplicationem aut diviſionem Arcus circularis. Poſteriores vero ſpectant ad aream Hyperbolæ. Et hæc Exempla ſufficiant ad illuſtrandam regulam hic adductam pro determinando indice. Ea deducitur ex Propoſitione quintâ: atque per eandem & regulas *Newtoni*, extractio radicum in Seriebus infinitis ad umbilicum perducitur, uti quivis facile percipiet qui mediocriter verſatus eſt in iis quæ hâc de re Authores hactenus protulere. Etenim facile demonſtrabitur Æquationem non admittere Seriem pro Radice quæ hâc methodo non eruitur. Sed hic non loquor de Seriebus quæ conflantur ex Terminis in quibus indeterminata x habet indices indeterminatos.

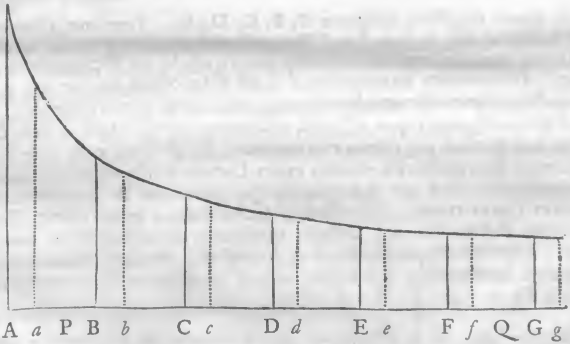
PARS

PARS SECUNDA.

DE

Interpolatione Serierum.

SIT recta quævis positione data PQ, super quam erigantur Ordinatæ quotcunque A, B, C, D, &c. sibi mutuo parallelæ, & æqualibus intervallis ab invicem distantes: designent autem hæ Ordinatæ Terminos Serici regularis, continue increscentes vel decrescentes, & eodem signo affectos; atque una eademque Curva transibit per extremitates omnium, quæ quidem determinabitur ex data



Æquatione ad Seriem, id est, ex datâ Æquatione generaliter exprimente relationem inter duas vel plures Ordinatas quasvis successivas.

Si ex data Æquatione illa differentiali erui poſſit Æquatio hujus Curvæ Algebraica, utpote quæ definit relationem inter Abſciſſas & Ordinatas correſpondentes, habebitur Ordinata quæcunque ex data ejus Abſciſſâ per reſolutionem Æquationum affectarum, adeoque Interpolatio Seriei abſoluta; quæ utique conſiſtit in aſſignatione cujuſlibet Termini primarii vel intermedii ex dato ejus loco in Serie. Ubi vero Æquatio Curvæ Algebraica nequit inveniri, id quod plerumque fit, nihil ulterius ſperandum eſt quam exhibere valorem Termini cujuſlibet quæſiti per Seriem convergentem, vel forte per Quadraturam Curvarum.

Hic autem loquor de Terminis æquidiſtantibus, quorum quippe relationes prodeunt ſcribendo numeros æquidifferentes ſucceſſive pro Abſciſſâ z in Æquatione differentiali Seriem definiente. Nam intervallum commune Ordinatarum ſuper Abſciſſam conſiſtentium, proportionale eſt incremento invariabili indeterminatæ z.

Propositio XVI.

Si idem ſit intervallum commune Terminorum primariorum & intermediorum, & detur unus ex intermediis, dabuntur omnes ex data Æquatione ad primarios.

In figurâ ſuperiore deſignent A, B, C, D, &c. Terminos primarios, & a, b, c, d, &c. intermedios; ſintque intervalla primariorum AB, BC, CD, &c. æqualia intervallis intermediorum ab, bc, cd, &c. quodque cuique. Dico omnes intermedios dari ex dato eorum quovis uno & ſimul Æquatione differentiali definiente relationem primariorum.

Etenim Æquatio quæ aſſignat relationem primariorum definit Curvam tranſeuntem per eorum extremitates; & Æquatio quæ definit relationem intermediorum definit etiam Curvam tranſeuntem per illorum extremitates. Sed per definitionem primariorum & intermediorum eadem Curva tranſit per amborum extremitates; quare eadem Æquatio quæ definit Curvam, definiet relationem Terminorum in utraque Serie. Et illa Æquatio, datur ex hypotheſi; adeoque datur lex continuandi intermedios, qui itaque omnes dabuntur ex dato quovis uno. Q. E. D.

Corollarium. Si in Æquatione quâvis differentiali exprimente relationem primariorum A, B, C, D, &c. valores ſucceſſivi Abſciſſæ z ſint PA, PB, PC, PD, &c. exiſtente puncto quovis P principio Abſciſſæ: tum

tum habebuntur relationes intermediorum ſcribendo pro z ſucceſſive $-\mathrm{P}a$, $\mathrm{P}b$, $\mathrm{P}c$, $\mathrm{P}d$, &c. in eâdem Æquatione. Nam Æquatio differentialis generaliter exprimit relationem inter duas quaſcunque Ordinatas ad certam diſtantiam ab invicem ſitas, ſive eæ cadant in Serie primariarum ſive in eâ intermediarum.

Exemplum I.

In progreſſione Terminorum Geometrica $1, x, x^2, x^3, x^4$, &c. ſi Termini in medio conſiſtentes inter primarios ſint a, b, c, d, e, &c. erit $b=ax$, $c=bx$, $d=cx$, $e=dx$, &c. quorum utique eadem eſt relatio ac primariorum.

Exemplum II.

Si Termini primarii ſint 1, 1, 2, 6, 24, 120, 720, &c. quorum relationes ſunt $\mathrm{B}=\mathrm{A}$, $\mathrm{C}=2\mathrm{B}$, $\mathrm{D}=3\mathrm{C}$, $\mathrm{E}=4\mathrm{D}$, &c. & Terminus directe in medio conſiſtens inter duos primos 1 & 1 ſit a; reliqui dabuntur per Æquationes $b=\frac{3}{2}a$, $c=\frac{5}{2}b$, $d=\frac{7}{2}c$, $e=\frac{9}{2}d$, &c. hi utpote conſiſtunt in medio inter duos quoſque primarios, ab iiſdem hinc inde æqualiter diſtantes.

Si vero a deſignet Terminum inter primum & ſecundum, cujus diſtantia a primo eſt tertia pars intervalli communis; pone $b=\frac{4}{3}a$, $c=\frac{7}{3}b$, $d=\frac{10}{3}c$, $e=\frac{13}{3}d$, &c. atque a, b, c, d, e, &c. erunt intermedii quorum quiſque diſtat tertia parte intervalli communis a primario ipſum antecedente, vel duabus tertiis partibus ejuſdem intervalli a primario ipſum immediate conſequente. Conſtat igitur relationem intermediorum dari ex Interpolatione relationis primariorum: & ea ſemper eſt interpolabilis, quoniam aſſignatur per Æquationem ad Seriem.

Exemplum III.

Proponatur Series $1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}$, &c. quæ producitur per multiplicationem continuam numerorum $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$, &c. & ſit a Terminus in medio inter primum & ſecundum, atque ponatur $b=\frac{2}{3}a$, $c=$

$c = \frac{4}{5} b$, $d = \frac{6}{7} c$, $e = \frac{8}{9} d$, &c. tum erunt b, c, d, e, &c. reliqui intermedii qui consistunt in medio inter duos quosque primarios.

SCHOLION.

Si Æquatio ad Seriem involvat tres Terminos, debent dari duo; & tres si involvat quatuor, & sic porro, ut habeantur reliqui intermedii. Hujus autem generis est Propositio *Newtoni* septima in Tractatu de quadratura Curvarum, quæ tamen non tantum obtinet in Curvis sed & in Serie quacunque. Atque usui venit hoc Theorema quotiescunque quæritur Terminus intermedius qui consistit prope initium Seriei: eo enim in casu ejus valor proveniet in Serie lentissime convergente; quare primo quærendus est respectivus intermedius satis remotus a principio, cujus utique valor prodibit in Serie cito approximante: dein ex hoc dato regrediendum est ad illum primo propositum per relationes Terminorum, ut in hâc Propositione.

PROPOSITIO XVII.

Omnis Series est interpolabilis, cujus Termini constant ex factoribus interpolabilibus.

Sit $A \times a \times \alpha$, $B \times b \times \beta$, $C \times c \times \gamma$, $D \times d \times \delta$, &c. Series cujus Termini conflantur ex tribus factoribus: dico eam esse interpolabilem, si interpolabiles sint tres factorum Series, scilicet A, B, C, D, &c. a, b, c, d, &c. & α, β, γ, δ, &c.

Quoniam enim Termini intermedii in Serie composita, eodem modo formantur ex intermediis correspondentibus in Seriebus simplicibus, ac primarii ex primariis; iidem invenientur ducendo respectivos intermedios Serierum simplicium in se mutuo. Ut si T sit Terminus inter B & C, t respectivus inter b & c, atque τ respectivus inter β & γ; tum Terminus correspondens in Serie composita, scilicet is inter $B \times b \times \beta$ & $C \times c \times \gamma$, erit factum sub illis tribus, nempe $T \times t \times \tau$. Et si plures vel pauciores sint factores, similiter demonstrabitur Propositio. Q. E. D.

Corollarium. Hinc in Serie interpolandâ, si duæ vel plures Series factorum sint accurate interpolabiles, possunt rejici ex computo: dein reliquæ sunt interpolandæ per methodos postea tradendas. Nam interpolatio non est temere suscipienda; sed ante exordium operis inquirendum est quænam sit Series simplicissima ex cujus intercalatione pendet ea Seriei propositæ.

ſitæ. Atque hæc præparatio eſt magna ex parte omnino neceſſaria, ut deveniamus ad concluſiones concinnas & elegantes.

Exemplum I.

Si Series intercalanda ſit $1, \frac{1}{6}x, \frac{3}{40}x^2, \frac{5}{112}x^3, \frac{35}{1152}x^4$, &c. primum reſolvo eandem in tres Series factorum ſimplicium ad modum ſequentem

$$x^0,\ x^1,\ x^2,\ x^3,\ x^4,\ \&c.$$

$$1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9},\ \&c.$$

$$1,\ \frac{1}{2},\ \frac{3}{8},\ \frac{5}{16},\ \frac{35}{128},\ \&c.$$

& harum trium Serierum palam eſt duas primas eſſe interpolabiles, non item tertiam; quæ tamen quum ſit ſimplicior, facilius interpolabitur quam ea quæ proponitur. Ut ſi deſideretur is in medio inter ſecundum & tertium Seriei compoſitæ; patet correſpondentes Terminos in prima & ſecunda Serierum ſimplicium eſſe $x^{\frac{3}{2}}$ & $\frac{1}{4}$ reſpective: in tertia Terminus correſpondens appelletur T, dein factum ſub hiſce tribus $x^{\frac{3}{2}}$, $\frac{1}{4}$, & T, ſive $\frac{1}{4}\,\mathrm{T}\,x\sqrt{x}$ erit is qui deſideratur. Adeoque Interpolatio Seriei compoſitæ reducitur ad Interpolationem Seriei ſimplicioris.

Exemplum II.

Si detur Series hujuſmodi $1, \frac{r}{p}\mathrm{A}, \frac{r+1}{p+1}\mathrm{B}, \frac{r+2}{p+2}\mathrm{C}, \frac{r+3}{p+3}\mathrm{D}$, &c. interpolare licet Series Numeratorum & Denominatorum ſeorſim, hoc eſt, Series

$$1,\ r,\ r.\overline{r+1},\ r.\overline{r+1}.\overline{r+2},\ \&c.$$

$$1,\ p,\ p.\overline{p+1},\ p.\overline{p+1}.\overline{p+2},\ \&c.$$

Deinde Terminus quilibet in Serie Numeratorum diviſus per reſpectivum in eâ Denominatorum, dabit correſpondentem in Serie propoſitâ. Si differentia inter r & p ſit numerus parvus, non opus eſt hujus artificii. Cæterum quando prædicta differentia eſt magna, neceſſe eſt interpolare Numeratores & Denominatores ſeorſim.

Scholion.

Huc etiam referri debent plurimæ hujusmodi præparationes. Exempli gratiâ, si Seriei

$$\&c.\ e,\ d,\ c,\ b,\ a,\ \ A,\ B,\ C,\ D,\ E,\ \&c.$$

hinc inde excurrentis in infinitum desideretur Terminus in medio inter duos medios primarios a & A. Duc Terminos primarios æqualiter a medio distantes in se mutuo; hoc est, A in a, B in b, &c. & componetur nova Series utrinque excurrens in infinitum

$$\&c.\ Dd,\ Cc,\ Bb,\ Aa,\ \ Aa,\ Bb,\ Cc,\ Dd,\ \&c.$$

cujus Termini a medio æqualiter hinc inde remoti sunt inter se æquales: & cujus Terminus in medio inter Aa & Aa erit quadratum Termini quæsiti qui consistit in medio inter a & A in Serie prius proposita. Igitur Terminus ille quæsitus erui possit per interpolationem alterutrius Seriei.

Notandum est Terminum desideratum consistere posse in diversis Seriebus, & ex eâ consideratione facilius nonnunquam inveniri. Ut si Terminus quilibet quæsitus lateat in medio inter primum & secundum in utrâvis Serierum sequentium

$$1,\ r,\ r.\overline{r+1},\ r.\overline{r+1}.\overline{r+2},\ \&c.$$

$$1,\ \frac{1}{p},\ \frac{1}{p.\overline{p+1}},\ \frac{1}{p.\overline{p+1}.\overline{p+2}},\ \&c.$$

Tum ducendo Terminos respectivos in se mutuo producetur nova Series

$$1,\ \frac{r}{p}A,\ \frac{r+1}{p+1}B,\ \frac{r+2}{p+2}C,\ \frac{r+3}{p+3}D,\ \&c.$$

inter cujus Terminum primum & secundum is qui medium locum tenet, æqualis est quadrato ejus primo propositi.

Nonnunquam etiam interpolatio prospere succedet per Logarithmos, præsertim si Terminorum permagnæ sint differentiæ. Sed hæc & similia addiscenda sunt experientiâ. Nam sicuti in Algebrâ vulgari tota ars Analystæ non consistit in resolutione Æquationum affectarum, sed in perducendo Problemata ad easdem; ita etiam in hac Analysi, minus solertiæ requiritur ad resolutionem Æquationum differentialium aut interpolationem Serierum; in inveniendis enim Seriebus quæ determinant quantitates incognitas, quæque idoneæ sint interpolationi, consistit longe major difficultas.

Propo-

Propositio XVIII.

Si Termini duarum Serierum formentur ducendo in se continue Fractiones quarum Numeratores & Denominatores increscant perpetuâ additione unitatis, & si iidem sint Numeratores in utrâque; Dico Terminum unius cujus distantia ab initio est differentia factorum in altera æqualem esse Termino hujus cujus distantia ab initio est differentia factorum in illa Serie, modo primi Termini sibi invicem æquentur.

Sint duæ Series quarum primi Termini A & a sibi mutuo æquentur

$$B = \frac{r}{p} A, \; C = \frac{r+1}{p+1} B, \; D = \frac{r+2}{p+2} C, \; E = \frac{r+3}{p+3} D, \text{ \&c.}$$

$$b = \frac{r}{q} a, \; c = \frac{r+1}{q+1} b, \; d = \frac{r+2}{q+2} c, \; e = \frac{r+3}{q+3} d, \text{ \&c.}$$

& in quibus iidem sunt Numeratores, dico Terminum prioris cujus distantia ab initio æqualis est differentiæ factorum in posteriore, nempe $q-r$, æqualem esse Termino posterioris, cujus distantia ab initio est $p-r$, scilicet differentiæ factorum in Serie priore. Et notandum, ubi $p-r$ vel $q-r$ est quantitas negativa, Terminos de quibus agitur consistere ante primos hisce intervallis.

Assumamus Terminum quemlibet Seriei prioris, verbi gratiâ septimum, positâ $A=1$, scilicet

$$G = 1 \times \frac{r}{p} \times \frac{r+1}{p+1} \times \frac{r+2}{p+2} \times \frac{r+3}{p+3} \times \frac{r+4}{p+4} \times \frac{r+5}{p+5}, \text{ \&c.}$$

Et primo si sit $p-r=0$, sive $p=r$; erit $p+1=r+1$, $p+2=r+2$, &c. adeoque omnes Numeratores & Denominatores sese mutuo destruunt, & restat $G=1$.

Si sit $p-r=1$, erit $p=r+1$; & inde $p+1=r+2$, $p+2=r+3$, &c. quo in casu evanescent omnes Numeratores præter primum, & Denominatores omnes præter ultimum; existente $G = 1 \times \frac{r}{p+5}$, vel $G = 1 \times \frac{r}{r+6}$ propter $p+5$ æqualem $r+6$.

Si

Si ſit $p-r=2$, vel $p=r+2$; erit $p+1=r+3$, $p+2=r+4$, &c. & nunc evaneſcent Numeratores omnes præter duos primos, & omnes Denominatores præter duos ultimos, manente $G=1\times\frac{r}{p+4}\times\frac{r+1}{p+5}$, vel $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}$, propter $p+4=r+6$, & $p+5=r+7$.

Et ſimiliter ſi ſit $p-r=3$, vel $p=r+3$; evaneſcent Numeratores omnes præter tres primos, & Denominatores omnes præter tres ultimos: & in illo caſu erit $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}\times\frac{r+2}{r+8}$. Atque univerſaliter in valore Termini G, tot ſemper erunt Numeratores totidemque Denominatores quot ſunt unitates in $p-r$, ut in Tabulâ ſequente

$$p-r=0,\quad G=1,$$

$$p-r=1,\quad G=1\times\frac{r}{r+6},$$

$$p-r=2,\quad G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7},$$

$$p-r=3,\quad G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}\times\frac{r+2}{r+8},$$

$$p-r=4,\quad G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}\times\frac{r+2}{r+8}\times\frac{r+3}{r+9},$$

&c.

Adeoque ſi ponatur $q=r+6$, ſive $q-r=6$; Terminus hujus Seriei 1, $\frac{r}{p}A$, $\frac{r+1}{p+1}B$, &c. cujus intervallum ab initio eſt $q-r$ ſive 6, æqualis eſt Termino hujus 1, $\frac{r}{q}a$, $\frac{r+1}{q+1}b$, &c. cujus intervallum ab initio eſt $p-r$. Et eodem modo manifeſta erit Propoſitio in aliis caſibus. Q. E. D.

Corollarium. Hinc ſi differentia factorum p & r non ſit permagna, Terminus Seriei prioris utcunque longe remotus ab initio ſemper determinabitur per Terminum in Serie poſteriore qui parum diſtat a principio. Ut Exemplis ſequentibus patebit.

EXEMPLUM

EXEMPLUM I.

Sit $r=3$, $p=5$, $q=10$; & hisce valoribus substitutis, evadent duæ Series

$$1,\ \frac{3}{5},\ \frac{3.4}{5.6},\ \frac{3.4.5}{5.6.7},\ \frac{3.4.5.6}{5.6.7.8},\ \&c.$$

$$1,\ \frac{3}{10},\ \frac{3.4}{10.11},\ \frac{3.4.5}{10.11.12},\ \frac{3.4.5.6}{10.11.12.13},\ \&c.$$

Est vero $q-r=7$, $p-r=2$; adeoque Terminus in priore cujus intervallum ab initio est 7, æqualis erit Termino in posteriore, cujus intervallum ab initio est 2; sive quod idem est, octavus Terminus Seriei prioris $\frac{3.4.5.6.7.8.9}{5.6.7.8.9.10.11}$ æqualis est tertio posterioris $\frac{3.4}{10.11}$. Et notandum est ubi differentia inter p & r est numerus integer, tum Terminum quemvis Seriei prioris semper æquari primario alicui in posteriore.

EXEMPLUM II.

Sit prior Series $1,\ \frac{2}{1}A,\ \frac{4}{3}B,\ \frac{6}{5}C,\ \frac{8}{7}D$, &c. & quoniam factorum incrementum est binarius, divide Numeratores & Denominatores per binarium; & prodibit Series $1,\ \frac{1}{\frac{1}{2}}A,\ \frac{2}{\frac{1}{2}+1}B,\ \frac{3}{\frac{1}{2}+2}C$, &c. ubi jam incrementum factorum est unitas; & per consequens hæc Series comparari possit cum illâ in Theoremate, prodeuntibus $r=1$, $p=\frac{1}{2}$: Sit etiam m intervallum inter primum Terminum Seriei & quemvis alium; & erit $m=q-r$, vel $m=q-1$, atque $q=m+1$; quo substituto evadet Series posterior

$$1,\ \frac{a}{m+1},\ \frac{2b}{m+2},\ \frac{3c}{m+3},\ \frac{4d}{m+4},\ \&c.$$

cujus Terminus ab initio remotus intervallo $p-r$, sive $-\frac{1}{2}$, æqualis erit Termino prioris cujus intervallum ab initio est quantitas quæcunque m. Hoc est, Terminus prioris utcunque longe remotus a principio, scilicet distantiâ utcunque magnâ m semper æquabitur Termino posterioris, qui consistit ante primum dimidio intervalli communis.

 Vel

Vel ſi ſumatur Series 1, $\frac{1}{2}$ A, $\frac{3}{4}$ B, $\frac{5}{6}$ C, $\frac{7}{8}$ D, &c. cujus Termini ſunt reciproci ejus de quâ nunc egimus, erit $r = \frac{1}{2}$, $p = 1$; & ſi m ſit intervallum inter Terminum primum & quemvis alium, erit $m = q - r$, vel $m = q - \frac{1}{2}$, ac $q = m + \frac{1}{2}$, & poſterior Series evadet

$$1, \frac{a}{2m+1}, \frac{3b}{2m+3}, \frac{5c}{2m+5}, \frac{7d}{2m+7}, \text{\&c.}$$

In qua Terminus ab initio remotus diſtantiâ $p - r$, ſive $\frac{1}{2}$; id eſt, Terminus in medio inter primum & ſecundum, æqualis erit Termino cuicunque prioris cujus diſtantia ab initio eſt quantitas quælibet m. Ut ſi deſideretur Terminus Seriei milleſimus primus, cujus utique intervallum ab initio eſt mille, erit $m = 1000$, & Terminus Seriei

$$1, \frac{a}{2001}, \frac{3b}{2003}, \frac{5c}{2005}, \frac{7d}{2007}, \text{\&c.}$$

qui conſiſtit iu medio inter primum 1 & ſecundum $\frac{1}{1999}$ æqualis erit milleſimo primo Termino hujus 1, $\frac{1}{2}$ A, $\frac{3}{4}$ B, $\frac{5}{6}$ C, $\frac{7}{8}$ D, &c. Cujus etiam Termini intermedii eodem modo inveniuntur; nam ſi pro m ſcribatur $999\frac{1}{2}$, proveniet Series

$$1, \frac{a}{2000}, \frac{3b}{2002}, \frac{5c}{2004}, \frac{7d}{2006}, \text{\&c.}$$

cujus Terminus in medio inter primum & ſecundum, æqualis eſt Termino hujus Seriei 1, $\frac{1}{2}$ A, $\frac{3}{4}$ B, $\frac{5}{6}$ C, $\frac{7}{8}$ D, &c. qui conſiſtit in medio inter milleſimum & milleſſimum primum.

EXEMPLUM III.

Si quæratur Terminus hujus Seriei 1, $\frac{2}{1}$ A, $\frac{5}{4}$ B, $\frac{8}{7}$ C, $\frac{11}{10}$ D, &c. cujus intervallum a principio ſit quantitas quælibet m: primo divide Numeratores & Denominatores per incrementum ipſorum commune 3, & evadet Series 1, $\frac{\frac{2}{3}}{\frac{1}{3}}$ A, $\frac{\frac{5}{3}}{\frac{4}{3}}$ B, &c. adeoque erit $r = \frac{2}{3}$, $p = \frac{1}{3}$, $q -$

$r =$

$r = q - \frac{2}{3} = m$; & inde $q = m + \frac{2}{3}$: unde Series poſterior evadet

$$1, \frac{2a}{3m+2}, \frac{5b}{3m+5}, \frac{8c}{3m+8}, \frac{11d}{3m+11}, \&c.$$

cujus Terminus ab initio diſtans intervallo $p - r$, ſive $-\frac{1}{3}$, id eſt, Terminus qui conſiſtit ante primum tertiâ parte intervalli communis, æqualis eſt Termino Seriei prioris, qui diſtat a principio, intervallo quantumvis magno m.

De Quantitatum Differentiis.

Sit a, b, c, d, e, Series quotcunque quantitatum, & ſi priores auferantur de poſterioribus, manebunt Differentiæ primæ $b-a$, $c-b$, $d-c$, $e-d$: dein ſi harum Differentiarum priores ſimiliter auferantur de poſterioribus, relinquentur Differentiæ ſecundæ $c-2b+a$, $d-2c+b$, $e-2d+c$; quarum rurſus Differentiæ conſtituunt Differentias tertias $d-3c+3b-a$, $e-3d+3c-b$, quantitatum a, b, c, d, e. Et ſic porro proceditur uſque dum perventum fuerit ad ultimam Differentiam ut in Tabulâ ſequente.

	1mæ	2dæ	3tiæ	4ta
a				
	$b-a$			
b		$c-2b+a$		
	$c-b$		$d-3c+3b-a$	
c		$d-2c+b$		$e-4d+6c-4b+a$
	$d-c$		$-3d+3c-b$	
d		$e-2d+$		
	$e-d$			
e				

Sit $1-x$ binomium in quo unciæ $+1$, -1 eædem ſint ac Coefficientes in Differentiis primis; tum unciæ quadrati $1-2x+x^2$, ſcilicet $+1$, -2, $+1$ erunt Coefficientes in Differentiis ſecundis; item unciæ Cubi 1, -3, $+3$, -1 erunt Coefficientes in Differentiis tertiis: & in genere Coefficientes in quolibet ordine Differentiarum erunt unciæ in correſpondente dignitate binomii. Atque hiſce præcognitis, pergere licet per ſaltum ad quemlibet ordinem Differentiarum abſque conſideratione intermediarum.

Duæ quantitates habent Differentiam primam, tres habent ſecundam, quatuor tertiam; nec ulteriores habere poſſunt. Sed nonnunquam accidit

cidit quendam ordinem Differentiarum conſtituere progreſſionem æqualium, quo in caſu ulteriores Differentiæ non habentur, utcunque magnus ſit quantitatum numerus. Sic in progreſſione arithmeticâ æquantur differentiæ primæ, adeoque non dantur ſecundæ. Et in Serie quadratorum 1, 4, 9, 16, 25, &c. quorum radices ſunt æquidifferentes, differentiæ primæ 3, 5, 7, 9, &c. ſunt in progreſſione arithmeticâ, ſecundæ ſunt æquales, câque de cauſâ tertiæ ſunt nihil. Sic etiam in Serie cuborum 1, 8, 27, 64, 125, 216, &c. Differentiæ primæ ſunt 7, 19, 37, 61, 91, &c. ſecundæ 12, 18, 24, 30, &c. tertiæ 6, 6, 6, &c. utique æquales; adeoque quartæ nihil.

Et univerſaliter ſint A, B, C, D, E, quantitates quotcunque datæ, z vero variabilis, tum in expreſſione $A + Bz + Cz^2 + Dz^3 + Ez^4$ ſcribe numeros quoſvis æquidifferentes ſucceſſive pro z; & quantitatum provenientium ultimæ Differentiæ determinabuntur per altiſſimam dignitatem z^4, nullâ ratione habitâ ad inferiores. Sic in hoc caſu quarta differentia eſt ultima, propterea quod quarta poteſtas z^4 ſit hic altiſſima.

Sæpiſſime differentiæ conſtituunt Seriem convergentem in caſibus quando non abrumpunt. Ut ſi a, b, c, d, e, &c. ſint fere inter ſe æquales, earumque Differentiæ primæ $b-a$, $c-b$, $d-c$, $e-d$, &c. etiam ſibi mutuo fere æquales, atque etiam ſecundæ ſimiliter & ſequentes ſibi mutuo quamproxime æquales, tum a, $b-c$, $c-2b+a$, $d-3c+3b-a$, &c. conſtituent Seriem convergentem. Item differentiæ Terminorum quorum relatio definitur Æquatione

$$T \times \overline{z^\theta + az^{\theta-1} + bz^{\theta-2} +} \text{ \&c.} = T' \times \overline{z^\theta + cz^{\theta-1} + dz^{\theta-2} +} \text{ \&c.}$$

ſumptæ ad modum præcedentem conficient Seriem convergentem. Sed non expectandum eſt Differentias quantitatum quarumcumque vel convergere vel abrumpere. Hoc tantum accidit in iis quantitatibus quæ increſcunt aut decreſcunt accurate vel quamproxime eâdem celeritate ac certæ quædam dignitates numerorum æquidifferentium.

De Deſcriptione Curvarum per data puncta.

Newtonus in Epiſtolâ ad *Oldenburgum* Anno 1676 datâ, poſtquam monſtraverat artificium evitandi Series nimium complexas in quadraturâ Curvarum trinomialium; inquit, "Sed hæc minoris facio, quod ubi "Series ſimplices non ſunt ſatis tractabiles, aliam nondum communi- "catam methodum habeo, quâ pro libitu acceditur ad quæſitum. "Ejus fundamentum eſt commoda, expedita, generalis ſolutio hujus "Problematis, *Curvam Geometricam deſcribere, quæ per data quotcunque* "*puncta tranſibit.* Docuit *Euclides* deſcriptionem Circuli per tria pun- "cta. Poteſt etiam Sectio Conica deſcribi per quinque data puncta:

"&

" & Curva trium dimensionum per septem data puncta; (adeo ut in
" potestate habeam descriptionem omnium Curvarum istius ordinis, quæ
" per septem tantum puncta determinantur.) Hæc statim Geometrice
" fiunt nullo calculo interposito. Sed superius Problema est alterius
" generis: & quamvis primâ fronte intractabile videatur; tamen res
" aliter se habet. Est enim fere ex pulcherrimis quæ solvere desi-
" derem.

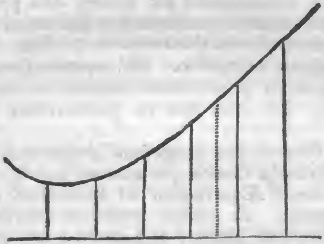
Newtonus in Propositione sexagesimâ *Arithmeticæ Universalis*, docet descriptionem Parabolæ Conicæ per quatuor puncta: vel potius docet methodum inveniendi Æquationem ad Parabolam quæ transibit per quatuor data puncta. Et eâdem methodo describere licet Lineam tertii ordinis per novem puncta, & Lineam quarti per quatuordecem. Et sic in reliquis. Sed nostrum institutum non requirit solutionem adeo generalem; sufficit enim describere figuram Parabolicam per extremitates quotcunque ordinatarum quæ sunt sibi mutuo, & etiam Axi Curvæ parallelæ. Sed neque descriptio Curvæ organica motu angulorum, aut alio quovis modo, ad præsens propositum conducit: eodem res redit sive Curva actu describatur, sive descripta concipiatur. Curvæ enim hic nullatenus sunt necessariæ, nisi quatenus adjumento sunt intellectui in rite concipiendo Problemate. Etenim descriptio Parabolæ per data puncta, idem est omnino Problema ac assignatio quantitatum ex datis ipsarum Differentiis; quæ per Algebram solam, idque per resolutionem Æquationum simplicium semper perficitur.

Propositio XIX.

Detur Series Ordinatarum æquidistantium ex una tantum parte pergens in infinitum, & oporteat invenire Lineam Parabolicam quæ transibit per extremitates omnium.

Designent A, A1, A2, A3, A4, &c. Ordinatas æquidistantes insistentes Abscissæ normaliter. Collige earum differentias primas B, B1, B2, B3, &c. secundas C, C1, C2, &c. tertias D, D1, &c. & sic porro. Adeo ut A sit Ordinata prima, B differentia prima duarum primarum Ordinatarum, C differentia secunda trium primarum Ordinatarum, D differentia tertia quatuor primarum & sic deinceps. Differentiæ autem colligi debent auferendo priores ubique de posterioribus; hoc est, ponendo $B = A1 - A$, $B1 = A2 - A1$, &c. $C = B1 - B$, &c. & ita porro in reliquis; ponendo eas esse negativas quæ oriuntur ex subductione quantitatis majoris de minore.

 Hisce



Hisce præmissis, sit T quævis Ordinata in genere, primaria vel intermedia, cujus distantia a primâ Ordinatâ A, nempe AT sit ad intervallum commune æquidistantium ut quantitas indeterminata z ad unitatem, eritque

$$\begin{aligned} T = A &+ \\ B \times \frac{z}{1} &+ \\ C \times \frac{z}{1} \times \frac{z-1}{2} &+ \\ D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} &+ \\ E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} &+ \\ F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} &+ \\ \&c. \end{aligned}$$

Hic est valor Ordinatæ cujusvis T jacentis ad easdem partes Ordinatæ primæ A, ad quas jacent reliquæ: quod si jaceret ad alteras, tum mutandum foret signum Abscissæ z. Etenim Abscissam pono affirmativam quæ ab initio tendit dextram versus inter Ordinatas posteriores; negativam vero quæ ad contrarias partes porrigitur. Propositio autem sic demonstratur.

Concipe Ordinatam T motu parallelo latam deferri ſuper Abſciſſam, adeo ut ſucceſſive deveniat ad loca reliquarum. Et quoniam illius diſtantia ab Ordinatâ primâ ponitur eſſe ad intervallum Ordinatarum commune ut z ad unitatem; erit z ſucceſſive æqualis 0, 1, 2, 3, 4, &c. & interea T æqualis Ordinatis A, A1, A2, A3, &c. per vices, cuique in ſuo loco. Igitur ad eruendas Coefficientes A, B, C, D, &c. quæ efficiunt Parabolam tranſire per extremitates Ordinatarum, in Æquatione ad figuram $T = A + B\frac{z}{1} + C\frac{z}{1} \times \frac{z-1}{2} + D\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} +$ &c. ſcribe Ordinatas A, A1, A2, A3, &c. ſucceſſive pro T; & interea pro z, longitudines Abſciſſæ ordine ſequentes, id eſt, 0, 1, 2, 3, &c. atque emergent Æquationes

$A = A,$	Unde $A = A,$
$A1 = A + B,$	$B = A1 - A,$
$A2 = A + 2B + C,$	$C = A2 - 2A1 + A,$
$A3 = A + 3B + 3C + 3D,$	$D = A3 - 3A2 + 3A1 - A,$
$A4 = A + 4B + 6C + 4D + E,$	$E = A4 - 4A3 + 6A2 - 4A1 + A;$
&c.	&c.

Quippe ex valoribus Ordinatarum A, A1, A2, &c. viciſſim eruuntur valores Coefficientium A, B, C, &c. Ex quibus patet Ordinatam primam A eſſe primam Coefficientem; item differentiam duarum primarum Ordinatarum eſſe Coefficientem ſecundam; & differentiam ſecundam primarum trium eſſe Coefficientem tertiam; & ſic in infinitum. Igitur valores Coefficientium in ſolutione poſiti efficiunt Parabolam tranſire per extremitates Ordinatarum. Q. E. D.

Idem aliter.

Fingamus eſſe in genere $T = A + B\frac{z}{1} + C\frac{z}{1} \times \frac{z-1}{2} + D\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} +$ &c. ubi A, B, C, D, &c. ſunt Coefficientes determinandæ. Scribe valores variabilium conſequentes T′, $z+1$ pro antecedentibus T, z; & emerget $T' = A + B\frac{z+1}{z} + C\frac{z+1}{1} \times \frac{z}{2} + D\frac{z+1}{1} \times \frac{z}{2} \times \frac{z-1}{3} +$ &c. de quo ſi ſubducatur valor ipſius T, habebitur

T′ =

$$T' - T = B + C\frac{z}{1} + D \times \frac{z}{1} \times \frac{z-1}{2} + \&c.$$

Ubi ſubſtituendo T'', T', $z+1$ pro T', T, z, prodibit $T'' - T' = B + C\frac{z+1}{1} + D\frac{z+1}{1} \times \frac{z}{2} + \&c.$ a quo valor ipſius $T' - T$ ſubductus relinquit

$$T'' - 2T' + T = C + D\frac{z}{1} + \&c.$$

Et ſimiliter invenies $T''' - 3T'' + 3T' - T = D + \&c.$ Deſignet jam T primam Ordinatam, & valor correſpondens Abſciſſæ erit nihil; quo ſubſtituto invenies

$$A = T,$$
$$B = T' - T,$$
$$C = T'' - 2T' + T,$$
$$D = T''' - 3T'' + 3T' - T,$$
&c.

Hoc eſt, prima Coefficiens A æqualis eſt Ordinatæ primæ T; ſecunda B æqualis eſt differentiæ inter duas primas Ordinatas T & T'; tertia C æqualis eſt differentiæ ſecundæ trium primarum Ordinatarum, T, T', T''; quarta C æqualis eſt differentiæ tertiæ primarum quatuor Ordinatarum; & ſic porro in reliquis ut jam demonſtratum eſt.

EXEMPLUM I.

Dentur quinque Ordinatæ 1, 4, 2, 3, 9, per quarum extremitates oportet Parabolam tranſire. Collige earum differentias primas 3, —2, 1, 6; ſecundas —5, 3, 5, tertias 8, 2; & ultimam —6. Dein juxta præſcripta in ſolutione Propoſitionis, pro A, B, C, &c. ponantur Ordinata prima, & prima quæque differentia reſpective; hoc eſt, $A=1$, $B=3$, $C=-5$, $D=8$, $E=-6$; at F, G, &c. erunt nihil. Hiſce autem valoribus ſubſtitutis, prodit Æquatio ad Parabolam

1	4	2	3	9
3	—2	1	6	
—5	3	5		
8	2			
—6				

$$T = 1 + 3\frac{z}{1} - 5\frac{z}{1} \times \frac{z-1}{2} + 8\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} - 6\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4},$$

quæ in ordinem reducta fit $T = \frac{12 + 116z - 111z^2 + 24z^3 - 3z^4}{12}$. Atque

que ad probandum opus, ſcribe 0, 1, 2, 3, 4, ſucceſſive pro Abſciſſa z, & pro T provenient Ordinatæ quinque propoſitæ.

Ceterum Ordinatæ poſſunt ſumi in ordine inverſo, modo mutentur ſigna differentiarum in ordinibus alternis: tunc ponendum eſt $A = 9$, $B = -6$, $C = 5$, $D = -2$, $E = -6$; quibus ſcriptis & Æquatione prodeunte in ordinem redacta, tandem obtinebitur $T =$

$\frac{108 - 92z + 9z^2 + 14z^3 - 3z^4}{12}$; in quâ ſi pro z ſubſtituantur 0, 1, 2, 3, 4, prodibunt Ordinatæ propoſitæ in ordine inverſo. Atque hic obtinentur duæ Æquationes pro eâdem Parabolâ, quoniam Abſciſſa initium ducit nunc a primâ, nunc ab ultimâ Ordinatâ.

EXEMPLUM II.

Oporteat nunc invenire Æquationem ad Parabolam quæ tranſit per extremitates ſex Ordinatarum æquidiſtantium 5, 3, 7, 23, 57, 115. Collige earum differentias primas, reliquaſque uſque dum perventum fuerit ad ultimas, ut in margine. Et invenies $A = 5$; $B = -2$, $C = 6$, $D = 6$. Quibus ſubſtitutis, oritur

5	3	7	23	57	115
−2	4	16	34	58	
6	12	18	24		
6	6	6			

$$T = 5 - 2\frac{z}{1} + 6\frac{z}{1} \times \frac{z-1}{2} + 6\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3},$$

quæ reducta fit $T = 5 - 3z + z^3$. Et ſi pro z ſcribas ſucceſſive 0, 1, 2, 3, 4, 5, provenient ſex Ordinatæ propoſitæ.

Linea recta tranſit per duo puncta, Parabola Conica per tria, Cubica per quatuor, Biquadratica per quinque, & ſic in infinitum. Nonnunquam vero accidit Curvam inferioris ordinis tranſire per plura puncta, ut in Exemplo noviſſimo. Ordo autem Parabolæ ſemper denotatur per ultimum Ordinem differentiarum. Quod ſi numerus Ordinatarum ſit infinitus, & progreſſio æqualium differentiarum non obveniat, in eo inquam caſu Curva erit dimenſionum infinitarum, valore ipſius T excurrente in Seriem infinitam.

SCHOLION.

In hac ſolutione poſuimus diſtantiam communem Ordinatarum eſſe unitatem; verum ſi pro eadem uſurpaſſem quantitatem quamvis n, prodiiſſet

$$T = A + B \times \frac{z}{n} + C \times \frac{z}{n} \times \frac{z-n}{2n} + D \times \frac{z}{n} \times \frac{z-n}{2n} \times \frac{z-2n}{3n} + \&c.$$

Finge jam Ordinatam secundam $A1 = A + \dot{A}n$,

tertiam - $A2 = A + 2\dot{A}n + \ddot{A}n^2$,

quartam $A3 = A + 3\dot{A}n + 3\ddot{A}n^2 + \dddot{A}n^3$,

quintam $A4 = A + 4\dot{A}n + 6\ddot{A}n^2 + 4\dddot{A}n^3 + \ddddot{A}n^4$,

&c.

Et invenies viciſſim $B = \dot{A}n$, $C = \ddot{A}n^2$, $D = \dddot{A}n^3$, $E = \ddddot{A}n^4$, &c. quibus valoribus ſubſtitutis pro B, C, D, E, &c. orietur

$$T = A + \dot{A}\frac{z}{1} + \ddot{A}\frac{z}{1} \times \frac{z-n}{2} + \dddot{A}\frac{z}{1} \times \frac{z-n}{2} \times \frac{z-2n}{3} + \&c.$$

Evadat jam intervallum commune n nihil, & $\dot{A}$, $\ddot{A}$, $\dddot{A}$, &c. evadent fluxiones Ordinatæ primæ A, dummodo fluxio Abſciſſæ z ſit unitas; atque proveniet

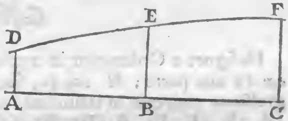
$$T = A + \dot{A}z + \frac{1}{2}\ddot{A}z^2 + \frac{1}{6}\dddot{A}z^3 + \frac{1}{24}\ddddot{A}z^4 + \&c.$$

Igitur coincidentibus Ordinatis æquidiſtantibus, incidimus in Seriem ubi Coefficientes Terminorum ſunt fluxiones Ordinatæ primæ reſpective diviſæ per numeros 1, 2, 6, 24, &c. qui generantur per continuam multiplicationem horum, 1, 2, 3, 4, &c. Et hoc primus deprehendit D. *Taylor* in *Methodo Incrementorum*, & poſtea *Hermanus* in Appendice ad *Phoronomiam*.

Hinc ſi Ordinata Curvæ cujuſcunque reſolvatur in Seriem hujus formæ $A + Bz + Cz^2 + Dz^3 + \&c.$ ubi exponentes Abſciſſæ z ſunt numeri integri & affirmativi; primus Terminus A eſt prima Ordinata, utique quæ tranſit per initium Abſciſſæ; primi duo $A + Bz$ denotant lineam rectam quæ tranſit per duo Curvæ puncta coincidentia, quæque proinde tangit Curvam; primi tres Termini $A + Bz + Cz^2$ definiunt Parabolam Conicam quæ tranſit per tria Curvæ puncta coincidentia, quæque eâ de cauſâ tangit Curvam, eandemque habet Curvaturam in puncto per quod tranſit Ordinata prima; primi quatuor Termini $A + Bz + Cz^2 + Dz^3$ definiunt Parabolam Cubicam quæ tranſit per quatuor Curvæ puncta coeuntia, id eſt, quæ tangit Curvam, eandemque habet Curvaturam & Variationem Curvaturæ in puncto Contactus. Denique tota Series $A + Bz + Cz^2 + Dz^3 + \&c.$ eſt Ordinata Parabolæ dimenſionum infinitarum quæ tangit Curvam, & in puncto Contactus, habet eandem Curvaturam, Variationem Curvaturæ,

Variationem Variationis, & ſic in infinitum, ut exprimitur a *Newtono* in Propoſitione decimâ Libri ſecundi Principiorum. Vel quod eodem redit, tota Series eſt Ordinata Parabolæ tranſeuntis per Ordinatas Curvæ æquidiſtantes, numero infinitas, & coincidentes cum Ordinatâ primâ.

Hinc ideam habemus analogiæ quæ eſt inter Methodum Differentialem & Methodum vulgarem Serierum; hæc procedit per Fluxiones ſive rationes differentiarum ultimas, illa vero generaliter per differentias cujuſcunque magnitudinis.



Deſignet DEF Curuam quamvis, cujus Abſciſſa AC decuſſat Ordinatas æquidiſtantes AD, BE, CF, ad angulos rectos. Sitque $AB = z$, $AD = A$; eritque ex ſeperioribus

$BE = A + \dot{A}z + \frac{1}{2}\ddot{A}z^2 + \frac{1}{6}\dddot{A}z^3 + \frac{1}{24}\ddddot{A}z^4 +$ &c. ſcilicet hicce valor ipſius $\dot{B}E$ eſt Ordinata Curvæ Parabolicæ quæ coincidit cum altera Curva in puncto D: igitur pro Area Curvæ uſurpare licet Aream ejuſdem Parabolæ, quæ per methodum Fluxionum inverſam prodit $ABED = Az + \frac{1}{2}\dot{A}z^2 + \frac{1}{6}\ddot{A}z^3 + \frac{1}{24}\dddot{A}z^4 + \frac{1}{120}\ddddot{A}z^5 +$ &c.

Et eâdem prorſus ratione ſi BE dicatur y, exiſtente $AB = BC = z$, erit Area

$$BCFE = yz + \frac{1}{2}\dot{y}z^2 + \frac{1}{6}\ddot{y}z^3 + \frac{1}{24}\dddot{y}z^4 + \frac{1}{120}\ddddot{y}z^5 + \text{\&c.}$$

In quâ ſi mutetur ſignum Abſciſſæ z, obtinebitur Area BEDA negative expreſſa, ſcilicet mutando ſignum Abſciſſæ obtinetur Area jacens ad alteras partes Ordinatæ. Sed Area illa affirmative expreſſa evadit

$$BEDA = yz - \frac{1}{2}\dot{y}z^2 + \frac{1}{6}\ddot{y}z^3 - \frac{1}{24}\dddot{y}z^4 + \frac{1}{120}\ddddot{y}z^5 - \text{\&c.}$$

Et hæc eſt Series D. *Johannis Bernoullii* exhibens Aream per Ordinatam ultimam ejuſque Fluxiones expreſſam; quamque nos jam dedimus per Fluxiones Ordinatæ primæ. Notandum autem eſt Seriem priorem non extendi ad caſus in quibus Ordinata prima tangit Curvam, & Seriem D. *Bernoullii* non extendi ad eos in quibus Ordinata ultima tangit Curvam. Nam Parabola cujus Area uſurpatur pro Area Curvæ quadrandæ nullam Ordinatarum tangere poteſt; adeoque coincidere nequit cum alterâ Curvâ Ordinatam ſuam tangente. Hujuſmodi enim Expreſſiones pro Areis & Ordinatis Curvarum præſupponunt formam Seriei $A + Bz + Cz^2 +$ &c. in quâ exponentes Abſciſſæ z ſunt numeri integri & affirmativi.

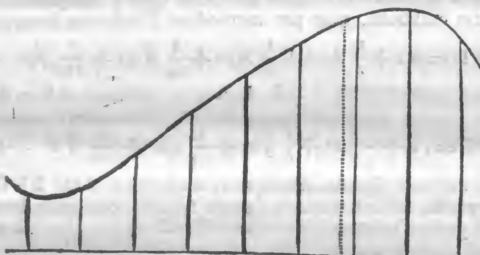
PROPO-

Propositio XX.

Detur Series Ordinatarum æquidistantium utrinque excurrens in infinitum, & oporteat invenire Lineam Parabolicam quæ transibit per extremitates omnium.

Casus primus.

Designet a Ordinatam in medio omnium, sintque $a2$, $a4$, $a6$, $a8$, &c. eæ ex una parte; & $2a$, $4a$, $6a$, $8a$, &c. eæ ex altera; pergente progressione utrinque in infinitum. Collige earum differentias primas 7B, 5B, 3B, 1B, B1, B3, B5, B7; secundas $6b$, $4b$, $2b$, b, $b2$, $b4$, $b6$; tertias 5C, 3C, 1C, C1, C3, C5; quartas $4c$, $2c$, c, $c2$, $c4$; & sic porro in reliquis, auferendo semper antecedentes de consequentibus ut in Propositione superiore.



$8a$ $6a$ $4a$ $2a$ a $a2$ T$a4$ $a6$ $a8$

7B 5B 3B 1B B1 B3 B5 B7

$6b$ $4b$ $2b$ b $b2$ $b4$ $b6$

5C 3C 1C C1 C3 C5

$4c$ $2c$ c $c2$ $c4$

3D 1D D1 D3

$2d$ d $d2$

1E E1

e

Sint jam a, b, c, d, e, &c. Ordinata media & differentiæ in ordinibus alternis respective. Sintque 1B & B1, 1C & C1, 1D & D1, 1E & E1, &c. duæ mediæ differentiæ in reliquis ordinibus; & ponantur B =

1B +

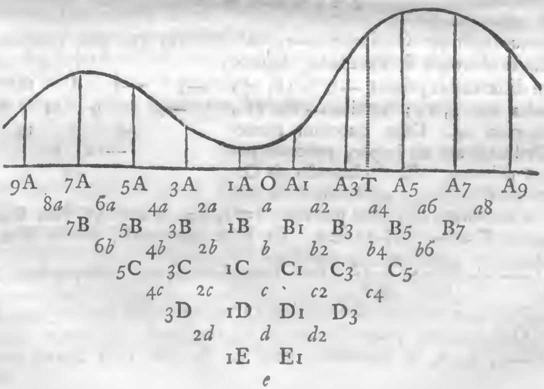
$1B + B1$, $C = 1C + C1$, $D = 1D + D1$, $E = 1E + E1$, &c. Sitque intervallum inter Ordinatam quamvis T & mediam a ad intervallum commune æquidiſtantium ut z ad unitatem; eritque Ordinata

$$T = a + \frac{Bz + bzz}{1.2} +$$
$$\frac{2Cz + czz}{1.2} \times \frac{zz - 1}{3.4} +$$
$$\frac{3D + dzz}{1.2} \times \frac{zz - 1}{3.4} \times \frac{zz - 4}{5.6} +$$
$$\frac{4E + ezz}{1.2} \times \frac{zz - 1}{3.4} \times \frac{zz - 4}{5.6} \times \frac{zz - 9}{7.8} +$$
$$\frac{5F + fzz}{1.2} \times \frac{zz - 1}{3.4} \times \frac{zz - 4}{5.6} \times \frac{zz - 9}{7.8} \times \frac{zz - 16}{9.10} +$$
&c.

Ubi notandum eſt Abſciſſam z eſſe negativam quando Ordinata quæſita T jacet ad contrarias partes Ordinatæ mediæ.

Caſus ſecundus.

Sint jam $1A$ & $A1$, Ordinatæ duæ mediæ; $A3$, $A5$, $A7$, $A9$, &c. eæ ex unâ parte; & $3A$, $5A$, $7A$, $9A$, &c. eæ ex alterâ. Collige earum differentias primas $8a$, $6a$, $4a$, $2a$, a, $a2$, $a4$, $a6$, $a8$; ſecundas $7B$, $5B$, $3B$, $1B$, $B1$, $B3$, $B5$, $B7$; tertias $6b$, $4b$, $2b$, b, $b2$, $b4$, $b6$, &c. & ſic deinceps; ſubducendo ubique priores de poſterioribus



Excerpe nunc differentias medias a, b, c, d, e, &c. ut & duas medias in aliis ordinibus, scilicet 1A & A1, 1B & B1, 1C & C1, 1D & D1, 1E & E1, &c. & ponantur $A = 1A + A1$, $B = 1B + B1$, $C = 1C + C1$, $D = 1D + D1$, $E = 1E + E1$, &c. sit vero O punctum in medio inter Ordinatas duas medias 1A & A1. Et cujusvis Ordinatæ T distantia a medio puncto, nempe OT sit ad intervallum commune æquidistantium ut z ad Binarium: Eritque

$$T = \frac{A + az}{2} +$$

$$\frac{3B + bz}{2} \times \frac{zz - 1}{4.6} +$$

$$\frac{5C + cz}{2} \times \frac{zz - 1}{4.6} \times \frac{zz - 9}{8.10} +$$

$$\frac{7D + dz}{2} \times \frac{zz - 1}{4.6} \times \frac{zz - 9}{8.10} \times \frac{zz - 25}{12.14} +$$

$$\frac{9E + ez}{2} \times \frac{zz - 1}{4.6} \times \frac{zz - 9}{8.10} \times \frac{zz - 25}{12.14} \times \frac{zz - 49}{16.18} +$$

&c.

Et in hoc quoque casu z est affirmativa quando T jacet ad easdem partes puncti medii O ac in Schemate, & negativa quando jacet ad contrarias. Uterque autem casus facillime demonstratur ad modum Propositionis superioris.

Exemplum I.

Dentur quinque Ordinatæ -3, -8, 1, 12, 37, per quarum extremitates ducenda sit Parabola. Quære earum differentias primas $-5, 9, 11, 25$; secundas $14, 2, 14$; tertias $-12, 12$; & ultimam 24. Dein quoniam numerus Ordinatarum est impar, procede per Casum primum. Et inchoando ab Ordinata media, perge ad differentias medias in ordinibus alternis, ponendo $a = 1$, $b = 2$, $c = 24$; dein $B = 9 + 11 = 20$, $C = -12 + 12 = 0$. Et hisce substitutis, prodit $T = 1 + \frac{20z + 2zz}{1.2} + \frac{24zz}{1.2} \times \frac{zz - 1}{3.4}$, sive $T = 1 + 10z + z^4$. Estque hæc Ordinata Parabolæ quatuor dimensionum transeuntis per extremitates quinque Ordinatarum propositarum, ut constabit scribendo numeros -2, -1, 0, 1, 2, successive pro z. In hoc casu Curva decussat

−3		−8		1		12		37
	−5		9		11		25	
		14		2		14		
			−12		12			
				24				

basin,

baſin, quoniam Ordinatæ ſunt partim negativæ & partim affirmativæ.

Exemplum II.

Dentur jam ſex Ordinatæ 1, 5, 10, 10, 5, 1 per quarum extremitates oportet ducere Parabolam. Quære earum differentias ut in margine: & quoniam numerus Ordinatarum eſt par, adhibeatur Caſus ſecundus. Tum incipiendo ab Ordinatis duabus mediis, & progrediendo ad binas medias differentias, erit A=10+10=20, B=−5−5=−10, C=6+6=12; deinde $a=0$, $b=0$, $c=0$.

$$\begin{array}{ccccccccccc}
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
& 4 & & 5 & & 0 & & -5 & & -4 & \\
& & 1 & & -5 & & -5 & & 1 & & \\
& & & -6 & & 0 & & 6 & & & \\
& & & & 6 & & 6 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

Quibus valoribus ſubſtitutis, erit $T = \frac{20}{2} - \frac{30}{2} \times \frac{zz-1}{4.6} + \frac{60}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10}$; quæ Æquatio in ordinem reducta evadit $T = \frac{689 - 50z^2 + z^4}{64}$. Atque ad probandam operationem, ſcribe −5, −3, −1, 1, 3, 5, ſucceſſive pro z, & exibunt Ordinatæ propoſitæ. Etenim in caſu Propoſitionis ſecundo, intervallum commune Ordinatatarum, ſive, quod perinde eſt, incrementum Abſciſſæ æquatur Binario.

In hoc Exemplo deſunt dignitates Abſciſſæ imparium dimenſionum, quoniam Ordinatæ a principio Abſciſſæ hinc inde æqualiter diſtantes ſunt ejuſdem ſigni, & inter ſe æquales. Nam hoc in caſu Æquatio ad Parabolam eadem manet, etiamſi mutetur ſignum Abſciſſæ. Quod ſi Ordinatæ propoſitæ fuiſſent +1, −5, +10, −10, +5, −1, vel +1, +5, +10, −10, −5, −1, ubi Ordinatæ a medio æqualiter remotæ ſunt etiamnum æquales, at diverſis ſignis affectæ; in eo inquam caſu defuiſſent dignitates Abſciſſæ parium dimenſionum.

Scholion.

De deſcriptione Curvæ Parabolici generis per data quotcunque puncta, egerunt plures celebres Geometræ poſt *Newtonum.* Sed omnes eorum ſolutiones eædem ſunt cum hiſce jam exhibitis; quæ quidem a *Newtonianis* vix diſcrepant, uti conſtabit ex Lemmate quinto Libri tertii *Principiorum* & *methodo Differentiali* a D. *Jones* edita. *Newtonus* quidem deſcribit Parabolam per data puncta; alii conſideraverunt aſſignationem Terminorum ex datis differentiis; ſed quocunque modo concipiatur, ſub quâcunque formâ proferatur, idem eſt Problema. Et

ſane inventio formarum quas habent valores Ordinatæ T, eſt perquam ingenioſa & ſummo Auctore digna: at poſtquam habentur formæ, inveſtigatio Problematis eſt facilis, in quam utique nihil aliud requiritur quam reſolutio Æquationum ſimplicium.

Sed notandum eſt formam Ordinatæ $A + Bz + Cz^2 + Dz^3 +$ &c. ex poteſtatibus compoſitæ, quam aſſumit *Newtonus* in demonſtrando ſuæ methodi fundamento, male huic negotio deſtinatam eſſe. Nam valor cujuſque Coefficientis prodit in Serie infinita; ſi quis vero aſſumſerit formas hic uſurpatas, incidet minimo labore in concluſiones ſuperiores.

PROPOSITIO XXI.

Datâ Serie Terminorum primariorum invenire intermedios, qui non longe diſtant a principio.

Super rectam poſitione datam ad anglos rectos, & in æqualibus ab invicem diſtantiis erigantur Ordinatæ reſpective æquales Terminis primariis; dein per Propoſitiones duas ſuperiores quæratur Linea Parabolica quæ tranſit per omnium extremitates, & eadem quoque tranſibit per extremitates intermediorum; qui itaque dabuntur ex data Æquatione ad Parabolam. Q. E. I.

EXEMPLUM I.

Interpolanda ſit Series $1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \frac{63}{256}$, &c. cujus Termini producuntur per multiplicationem continuam numerorum $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$, &c. quære differentias Terminorum & differentias differentiarum, ut ſequitur.

$$1 \quad \frac{1}{2} \quad \frac{3}{8} \quad \frac{5}{16} \quad \frac{35}{128} \quad \frac{63}{256} \text{ \&c.}$$
$$-\frac{1}{2} \quad -\frac{1}{8} \quad -\frac{1}{16} \quad -\frac{5}{128} \quad -\frac{7}{256}$$
$$\frac{3}{8} \quad \frac{1}{16} \quad \frac{3}{128} \quad \frac{3}{256}$$
$$-\frac{5}{16} \quad -\frac{5}{128} \quad -\frac{7}{256}$$
$$\frac{35}{128} \quad \frac{7}{256}$$
$$-\frac{63}{256}$$

Quoniam

Quoniam hæc Series ex unâ tantum parte excurrit in infinitum, peragenda est Interpolatio per Propositionem decimam nonam. Et si primus Terminus 1 adhibeatur pro Ordinata prima, erit $A = 1$, $B = -\frac{1}{2}$, $C = +\frac{3}{8}$, $D = -\frac{5}{16}$, $E = +\frac{35}{128}$, &c. quibus substitutis, prodit

$$T = 1 - \frac{1}{2} \times \frac{z}{1} + \frac{3}{8} \times \frac{z}{1} \times \frac{z-1}{2} - \frac{5}{16} \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + \&c.$$

$$\text{id est, } T = 1 - \frac{1}{2} A \frac{z}{1} - \frac{3}{4} B \frac{z-1}{2} - \frac{5}{6} C \frac{z-2}{3} - \frac{7}{8} D \frac{z-3}{4} - \&c.$$

Ubi jam A, B, C, D, &c. designant Terminos hujus Seriei, more *Newtoni*. Ceterum Terminus quilibet primarius usurpari potest pro Ordinata prima, eritque $A = \frac{1}{2}$, $B = -\frac{1}{8}$, $C = +\frac{1}{16}$, $D = -\frac{5}{128}$, $E = +\frac{7}{256}$, &c. adeoque

$$T = \frac{1}{2} - \frac{1}{8} \times \frac{z}{1} + \frac{1}{16} \times \frac{z}{1} \times \frac{z-1}{2} - \&c.$$

$$\text{id est, } T = \frac{1}{2} - \frac{1}{4} A \frac{z}{1} - \frac{3}{6} B \frac{z-1}{2} - \frac{5}{8} C \frac{z-2}{3} - \frac{7}{10} D \frac{z-3}{4} - \&c.$$

Sed sciendum est z esse distantiam inter Terminum quæsitum & illum qui adhibetur pro Ordinata prima. Ut si Terminus desideratus T stet in medio inter primum & secundum, pone $+\frac{1}{2}$ pro z in priore valore ipsius T; & $-\frac{1}{2}$ pro eadem in posteriore valore; atque pro eodem Termino habebis duas Series sequentes

$$1 - \frac{1}{4} A + \frac{3}{16} B + \frac{15}{36} C + \frac{35}{64} D + \frac{63}{100} E + \&c.$$

$$\frac{1}{2} + \frac{1}{8} A + \frac{9}{24} B + \frac{25}{48} C + \frac{49}{80} D + \frac{81}{120} E + \&c.$$

Et quoniam hæ Series convergunt quam lentissime, summandæ sunt per Theorema in Scholio Propositionis decimæ primæ. Atque sciendum est valores Terminorum prodire simplicissimos quando Terminus qui consistit proximus intermedio quæsito, adhibetur pro Ordinata prima. Sed quando Terminus quæsitus longissime distat ab initio, usui venit Propositio decima octava, ut monstrabitur in sequentibus.

Exemplum II.

Interpolanda ſit Series 1, 1, 2, 6, 24, 120, 720, &c. cujus Termini generantur per multiplicationem continuam numerorum 1, 2, 3, 4, 5, &c. Quoniam hi Termini increſcunt celerrime, eorum differentiæ conficient progreſſionem divergentem, id quod impedit quo minus Ordinata Parabolæ accedat ad veritatem. In hoc igitur & ſimilibus caſibus interpolo Logarithmos Terminorum, quorum utique differentiæ conſtituere queunt Seriem celeriter convergentem, etiamſi ipſi Termini increſcunt celerrime ut in Exemplo præſente.

Propono jam inventionem Termini qui conſiſtit in medio inter duos primos 1 & 1. Et quoniam Logarithmi Terminorum initialium habent differentias lente convergentes, primum quæram Terminum in medio conſiſtentem inter duos ab initio ſatis remotos, verbi gratiâ, inter decimum primum 3628800 & decimum ſecundum 39916800: & ex eo dato regrediar ad Terminum quæſitum per Propoſitionem decimam ſextam. Quumque habeantur aliquot Termini conſiſtentes ex utrâque parte intermedii qui primo eruendus eſt, inſtituo operationem per Caſum ſecundum Propoſitionis vigeſimæ. Etenim ubi computus non eſt in ſpeciebus, ſed in numeris, procedere licet hâc methodo quotieſcunque datur ſatis amplus Terminorum numerus ex utraque parte Termini quæſiti conſiſtentium, etiamſi ipſa Series interpolanda reapſe non excurrat utrinque in infinitum.

Excerpo nunc ex Tabula Logarithmos duodecim Terminorum, quorum primus eſt is Termini ſexti 120; ſic ut habeantur ſex ante, totidemque poſt illum qui quæritur. Dein quoniam Terminus ille deſideratus conſiſtit directe in medio omnium, erit in Caſu ſecundo Propoſitionis vigeſimæ primæ Abſciſſa $z = 0$; & propterea differentiæ primæ, tertiæ, reliquæque ordinum imparium quæ in z multiplicantur, non ingredientur computum; igitur tantummodo colligo differentias ſecundas, quartas, ceteraſque in ordinibus paribus ut vides

Logarithmi

Logarithmi	Diffe. 2dæ	quartæ	ſextæ	octavæ	decimæ
2.0791812460					
2.8573324964	669467896				
3.7024305364	579919470	21154180			
4.6055205234	511525224	14443928	2568588		
5.5597630329	457574906	10302264	1446210	541511	
6.5597630329	413926852	7606810	865343	259252	156590
7.6011557180	377885608	5776699	543728	133583	65082
8.6803369641	347621063	4490316	355696	72996	
9.7942803164	321846834	3559629	240660		
10.9404083521	299632234	2869602			
12.1164996111	280287236				
13.3206195938					

Excerpo jam binos medios Logarithmos, binaſque medias differentias; eorumque Summas pono æquales A, B, C, D, &c. reſpective ut vides

$$\begin{array}{r}6.5597630329\\7.6011557180\\\hline A=14.1609187509,\end{array}\qquad\begin{array}{r}413926852\\377885608\\\hline B=791812460,\end{array}\qquad\begin{array}{r}7606810\\5776699\\\hline C=13383509,\end{array}$$

$$\begin{array}{r}865343\\543728\\\hline D=1409071,\end{array}\qquad\begin{array}{r}259252\\133583\\\hline E=392835,\end{array}\qquad\begin{array}{r}156590\\65082\\\hline F=221672,\end{array}$$

Tum in Caſu ſecundo Propoſitionis vigeſimæ primæ; ſubſtituo o pro z, & habeo

$$T=\frac{1}{2}A-\frac{1}{16}B+\frac{3}{256}C-\frac{5}{2048}D+\frac{35}{65536}E-\frac{63}{524288}F+\&c.$$

Scribe jam ipſorum A, B, C, D, E, F, valores mox repertos, & computus ſic ſe habebit

$$\begin{array}{r}7.08045937545\\1568380\\2098\\\hline +\,7.08047508023\end{array}\qquad\qquad\begin{array}{r}494882787\\34401\\266\\\hline -\,494917454\end{array}$$

Ac ſubducendo Summam Terminorum negativorum ab eâ affirmativorum, remanet T = 7.07552590569. Eſtque hic Logarithmus numeri 11899423.08, utique qui conſiſtit in medio inter Terminos 3628800 & 39916800.

Jam ſicuti Termini primarii formantur ducendo primum continue in numeros 1, 2, 3, 4, &c. ita per Propoſitionem decimam ſextam, intermedii generantur ducendo intermedium inter primum & ſecundum in numeros $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. continue. Verbi gratiâ productum ſub decem factoribus $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, $5\frac{1}{2}$, $6\frac{1}{2}$, $7\frac{1}{2}$, $8\frac{1}{2}$, $9\frac{1}{2}$, $10\frac{1}{2}$, & Termino qui conſiſtit in medio inter primum & ſecundum, æquale eſt intermedio mox invento 11899423.08, cujus utique locus eſt in medio inter decimum primum & decimum ſecundum. Igitur ſi Terminus ille intermedius dividatur per $10\frac{1}{2}$, & quotus per $9\frac{1}{2}$, & quotus noviſſimus per $8\frac{1}{2}$, & ſic porro uſque ad diviſorem $1\frac{1}{2}$; ultimus quotorum æqualis erit Termino in medio inter 1 & 1. Termini vero intermedii per diviſionem illam prodeuntes ſunt

Primarii	Intermedii
39916800	11899423.08
3628800	1133278.389
362880	119292.4620
40320	14034.40729
5040	1871.254305
720	287.8852777
120	52.34277777
24	11.63172839
6	3.323350969
2	1.329340388
1	.8862269251
1	1.7724538502

Unde conſtat Terminum inter 1 & 1 eſſe .8862269251; cujus quadratum eſt .7853... &c. ſcilicet Area Circuli cujus Diameter eſt unitas. Atque illius Termini duplus 1.7724538502, ſcilicet qui conſiſtit antę primum primarium dimidio intervalli communis, æqualis eſt Radici quadrati numeri 3.1415926...., &c. qui denotat Circumferentiam Circuli.

Circuli, cujus Diameter eſt unitas. Etenim ſi quadrata primariorum conſtituant novam Seriem 1, 1, 4, 36, 576, 14400, &c. Terminus in medio inter primum & ſecundum erit ad unitatem ut Area Circuli ad quadratum circumſcriptum: & Terminus qui conſiſtit ante primum dimidio intervalli communis erit ad unitatem ut Circumferentia Circuli ad Diametrum. Ceterum in ſequentibus monſtrabitur hujuſmodi Series interpolari poſſe abſque Logarithmis.

Exemplum III.

Quadranda ſit Curva cujus Ordinata eſt $x^{\theta-1} \times \overline{e+fx^{\eta}}^{\lambda}$. Scribe 0, 1, 2, 3, &c. ſucceſſive pro indice λ, & proveniet Series Ordinatarum æquidiſtantium $x^{\theta-1} \times \overline{e+fx^{\eta}}^{0}$, $x^{\theta-1} \times \overline{e+fx^{\eta}}^{1}$, $x^{\theta-1} \times \overline{e+fx^{\eta}}^{2}$, &c. inter quas Ordinata propoſita locum obtinet intervallo λ diſtantem ab initio. Igitur Area quæſita eundem etiam obtinebit locum inter Areas illarum Ordinatarum, quæ conſtituunt progreſſionem ſequentem Terminorum æquidiſtantium,

$$\frac{1}{\theta}x^{\theta},\ \frac{e}{\theta}x^{\theta} + \frac{f}{\theta+\eta}x^{\theta+\eta},\ \frac{e^2}{\theta}x^{\theta} + \frac{2ef}{\theta+\eta}x^{\theta+\eta} + \frac{f^2}{\theta+2\eta}x^{\theta+2\eta}, \text{ \&c.}$$

Jam ſi hæ Areæ interpolarentur per Propoſitionem decimam nonam, proveniret pro Areâ quæſitâ, eadem Series ac per methodum vulgarem reducendi Ordinatam in Seriem convergentem ut exinde inveniatur fluens. Attamen ſi Termini in Serie Arearum primo dividantur per Terminos hujus progreſſionis Geometricæ reſpective, nempe $\overline{e+fx^{\eta}}^{0}$, $\overline{e+fx^{\eta}}^{1}$, $\overline{e+fx^{\eta}}^{2}$, $\overline{e+fx^{\eta}}^{3}$, &c. Hoc eſt, ſi ponantur

$$A = x^{\theta} \times \frac{\frac{1}{\theta}}{\overline{e+fx^{\eta}}^{0}},$$

$$A1 = x^{\theta} \times \frac{\frac{e}{\theta} + \frac{fx^{\eta}}{\theta+\eta}}{\overline{e+fx^{\eta}}^{1}},$$

$$A2 = x^{\theta} \times \frac{\frac{e^2}{\theta} + \frac{2efx^{\eta}}{\theta+\eta} + \frac{f^2x^{2\eta}}{\theta+2\eta}}{\overline{e+fx^{\eta}}^{2}},$$

$$A3 = x^{\theta} \times \frac{\frac{e^3}{\theta} + \frac{3e^2fx^{\eta}}{\theta+\eta} + \frac{3ef^2x^{2\eta}}{\theta+2\eta} + \frac{f^3x^{3\eta}}{\theta+3\eta}}{\overline{e+fx^{\eta}}^{3}},$$

&c.

& instituatur computus juxta Propositionem decimam nonam, invenientur differentiæ

$$B = \frac{-\eta f x^{\theta+\eta}}{\theta.\overline{\theta+\eta}.\overline{e+fx^{\eta}}}, \quad C = \frac{+2\eta^2 f^2 x^{\theta+2\eta}}{\theta.\overline{\theta+\eta}.\overline{\theta+2\eta}.\overline{e+fx^{\eta}}\rvert^{2}}, \quad D = \frac{-6\eta^3 f^3 x^{\theta+3\eta}}{\theta.\overline{\theta+\eta}.\overline{\theta+2\eta}.\overline{\theta+3\eta}.\overline{e+fx^{\eta}}\rvert^{3}}, \text{ \&c.}$$

quibus substitutis pro A, B, C, D, &c. & λ pro z; invenies Terminum distantiâ λ ab initio remotum, esse

$$\frac{x^{\theta}}{\theta} - \frac{\lambda \eta f x^{\theta+\eta}}{\theta.\overline{\theta+\eta}.\overline{e+fx^{\eta}}} + \frac{\lambda.\overline{\lambda-1}.\eta^2 f^2 x^{\theta+2\eta}}{\theta.\overline{\theta+\eta}.\overline{\theta+2\eta}.\overline{e+fx^{\eta}}\rvert^{2}} - \frac{\lambda.\overline{\lambda-1}.\overline{\lambda-2}.\eta^3 f^3 x^{\theta+3\eta}}{\theta.\overline{\theta+\eta}.\overline{\theta+2\eta}.\overline{\theta+3\eta}.\overline{e+fx^{\eta}}\rvert^{3}} + \text{\&c.}$$

Sed quia Termini interpolandi divisi erant per dignitates ipsius $e+fx^{\eta}$, quisque scilicet per dignitatem cujus index erat distantia Termini ab initio; e contra duc Terminum mox inventum in dignitatem prædicti binomii, cujus index est λ, sua utique distantia ab initio, & pro Area Curvæ habebitur

$$\frac{x^{\theta}}{\theta} \times \overline{e+fx^{\eta}}\rvert^{\lambda} \text{ in } 1 - \frac{\lambda \eta f x^{\eta}}{\overline{\theta+\eta}.\overline{e+fx^{\eta}}} + \frac{\lambda.\overline{\lambda-1}.\eta^2 f^2 x^{2\eta}}{\overline{\theta+\eta}.\overline{\theta+2\eta}.\overline{e+fx^{\eta}}\rvert^{2}} - \text{\&c.}$$

Vel ponendo $y = \frac{fx^{\eta}}{e+fx^{\eta}}$, $r = \frac{\theta+\eta}{\eta}$; atque scribendo Seriem more *Newtoni*, prodibit

$$\frac{x^{\theta}}{\theta} \times \overline{e+fx^{\eta}}\rvert^{\lambda} - \frac{\lambda}{r} A y - \frac{\lambda-1}{r+1} B y - \frac{\lambda-2}{r+2} C y - \frac{\lambda-3}{r+3} D y - \text{\&c.}$$

pro Area Curvæ cujus Ordinata in genere est $x^{\theta-1} \times \overline{e+fx^{\eta}}\rvert^{\lambda}$. Et hæc Series transmutata per Propositionem septimam, migrabit in eam *Newtoni* pro quadraturâ Curvæ binomialis. Abrumpit quando index λ est integer & affirmativus; sed post debitam Ordinatæ præparationem abrumpet semper ubi Curva est quadrabilis. Ceterum ejus usus præcipuus est qnod exhibeat Areas in Serie admodum simplici. Si Coefficientes e, f, contrariis signis afficiantur, præferenda est Series *Newtoni*; & nostra ubi iisdem.

Exemplum IV.

Oporteat assignare Uncias Binomii ex data Uncia media in dignitate cujus index est numerus par. Si n denotet indicem dignitatis, & Uncia media ducatur continue in fractiones $\frac{n}{n+2}$, $\frac{n-2}{n+4}$, $\frac{n-4}{n+6}$, &c. facta erunt Unciæ reliquæ consistentes ex utraque parte mediæ.

$$a6 = a \times \frac{n}{n+2} \times \frac{n-2}{n+4} \times \frac{n-4}{n+6},$$

$$a4 = a \times \frac{n}{n+2} \times \frac{n-2}{n+4},$$

$$a2 = a \times \frac{n}{n+2},$$

$$a = a,$$

$$2a = a \times \frac{n}{n+2},$$

$$4a = a \times \frac{n}{n+2} \times \frac{n-2}{n+4},$$

$$6a = a \times \frac{n}{n+2} \times \frac{n-2}{n+4} \times \frac{n-4}{n+6},$$

Et ſi a deſignet Unciam mediam, tum $a2$, $a4$, $a6$, &c. ex una parte; & $2a$, $4a$, $6a$, &c. ex alterâ deſignabunt reliquas Uncias. Dein ineundo computum per caſum primum Propoſitionis vigeſimæ invenies

$$a - \frac{rr}{2.\overline{n+2}} A - \frac{rr-4}{4.\overline{n+4}} B - \frac{rr-16}{6.\overline{n+6}} C - \frac{rr-36}{8.\overline{n+8}} D - \&c.$$

eſſe Terminum Seriei interpolandæ, cujus diſtantia a Termino media a eſt ad intervallum commune primariorum ut r ad Binarium. Exempli gratiâ, in dignitate decimâ ſecundâ Unciæ ſunt 1, 12, 66, 220, 495, 792, 924, 792, &c. exiſtente Unciâ mediâ $a = 924$. Et ſi quæratur ea quæ ternario diſtat a media, erit $r = 6$; quo ſubſtituto, & 12 pro n; prodit

$$924 - \frac{36}{2.14} A - \frac{32}{4.16} B - \frac{20}{6.18} C,$$

pro Uncia quæſita, abrumpente Serie: & hi Termini liberati a fractionibus ſunt $925 - 1188 + 594 - 110$, quorum ſumma ſub propriis ſignis eſt 220, qui eſt valor Unciæ quæſitæ.

Et ad eundem modum ſi index n ſit numerus impar, & A altera e mediis Unciis; tum erit

$$A - \frac{rr-1}{2.\overline{n+3}} A - \frac{rr-9}{4.\overline{n+5}} B - \frac{rr-25}{6.\overline{n+7}} C - \frac{rr-49}{8.\overline{n+9}} D - \&c.$$

Uncia cujus diſtantia a puncto inter duas medias Uncias intermedio, eſt ad intervallum commune ut r ad Binarium.

Et hæ Series in permagna dignitate convergent dummodo intervallum inter Unciam mediam & quæſitam ſit exiguum reſpectu indicis poteſtatis.

SCHOLION.

Scholion.

Poſtquam Series interpolanda debite præparatur per Propoſitionem decimam ſeptimam; etiamſi eadem utrinque excurrat in infinitum procedere licet per Propoſitionem decimam nonam præterquam ubi Termini a medio hinc inde æqualiter diſtantes ſunt inter ſe æquales: ubi hoc accidit, adhibeatur caſus primus Propoſitionis vigeſimæ, ſi Terminus quivis primarius quodam jure vindicet locum in medio omnium: vel ſi duo Termini eodem jure locum medium vindicent, adhibeatur caſus ſecundus ejuſdem Propoſitionis. Et in aliis caſibus ad libitum fere proceditur.

Propositio XXII.

Datâ Serie Terminorum æquidiſtantium, invenire Terminum quemvis primarium vel intermedium utcunque longe diſtantem ab initio Seriei.

Si Terminus quæſitus longe removetur a principio, tum per Propoſitionem decimam octavam quære aliam Seriem in qua Terminus ille deſideratus conſtituet Terminum prope initium, dein procede ut in Propoſitione ſuperiore.

Exemplum I.

Proponatur invenire Terminum quemvis hujus Seriei, $1,\ \frac{2}{1}A,\ \frac{4}{3}B,\ \frac{6}{5}C,\ \frac{8}{7}D$, &c. a principio quantumvis magno intervallo m diſtantem. Per Propoſitionem decimam octavam, Terminus Seriei

$$1,\ \frac{a}{m+1},\ \frac{2b}{m+2},\ \frac{3c}{m+3},\ \frac{4d}{m+4},\ \&c.$$

qui conſiſtit ante primum dimidio intervalli communis, æqualis erit Termino Seriei prioris cujus intervallum ab initio eſt m. Conſtat autem per Exemplum ſecundum Propoſitionis vigeſimæ primæ, Terminum dimidio intervalli communis diſtantem ante primum in Serie Numeratorum, 1, 1.1, 1.1.2, 1.1.2.3, &c. i. e. in hac 1, 1, 2, 6, 24, 120, &c. eſſe Radicem quadratam Numeri 3.1415926 .... &c. Quare tantum interpolo Denominatores nempe

$$1,\ \frac{1}{m+1},\ \frac{1}{m+1.m+2},\ \frac{1}{m+1.m+2.m+3},\ \&c.$$

Et

Et quoniam hæc Series continuari poteſt utrinque in infinitum; actu continuabitur, & evadet

$$\&c.\ \overline{m-2.m-1.m},\ \overline{m-1.m},\ m,\ ,1,\ \frac{1}{m+1},\ \frac{1}{m+1.m+2},\ \&c.$$

Ubi Terminus quæſitus conſiſtit in medio inter duos medios m & 1: ſed quia differentiæ horum Terminorum ſunt permagnæ; ducantur ii a medio hinc inde æqualiter diſtantes in ſe mutuo; hoc eſt, m in 1; $\overline{m-1.m}$ in $\frac{1}{m+1}$; & ſic porro; & prodibit nova Series

$$\&c.\ \frac{m-2}{m+2}\cdot\frac{m-1}{m+1}\cdot m,\ \frac{m-1}{m+1}m,\ m,\ m,\ m\frac{m-1}{m+1},\ m\frac{m-1}{m+1}\cdot\frac{m-2}{m+2},\ \&c.$$

utrinque excurrens in infinitum, & habens Terminos inter ſe æquales qui æqualiter diſtant a medio. Sed & Terminus in medio conſiſtens inter duos medios primarios m & m eſt quadratum ejus in medio inter m & 1 in Serie priore. In noviſſimâ igitur Serie quæratur Terminus ille inter m & m per Caſum ſecundum Propoſitionis vigeſimæ, & idem invenietur

$$m+\frac{m}{4.m+1}+\frac{9m}{32.m+1.m+2}+\frac{75m}{128.m+1.m+2.m+3}+\&c.$$

Quam ducito in Circumferentiam Circuli, ſcilicet Terminum reſpectivum in Serie Numeratorum, & habebis pro quadrato Termini quæſiti

$$3.14159\ldots\&c.\ \text{in}\ m+\frac{A}{4.m+1}+\frac{9B}{8.m+2}+\frac{25C}{12.m+3}+\frac{49D}{16.m+4}+\&c.$$

Igitur Terminus Seriei propoſitæ $1,\ \frac{2}{1}A,\ \frac{4}{3}B,\ \frac{6}{5}C$, &c. diſtans intervallo m ab initio æqualis eſt mediæ proportionali inter Circumferentiam Circuli & Seriem illam; quæ eo citius convergit quo major eſt m, hoc eſt quo longius diſtat ab initio Terminus quæſitus.

Exempli gratiâ ſit $m=100$ & primus Terminus Seriei erit Circumferentia Circuli ducta in 100, ſive $A=314.15927$; dein erit $B=\frac{A}{4.101}=.77762$, $C=\frac{9B}{8.102}=.00858$, $D=\frac{25C}{12.103}=.00017$; atque ſumma horum quatuor Terminorum eſt 314.94564, cujus Radix quadrata 17.7467079 eſt Terminus centeſimus primus Seriei interpolandæ; ſive productum ſub factoribus $\frac{2}{1}\times\frac{4}{3}\times\frac{6}{5}\times\frac{8}{7}$, &c. quorum numerus eſt Centum. Et eâdem ratione invenire licet Terminum quemvis intermedium:

$$\begin{array}{r} 314.15927 \\ 77762 \\ 858 \\ 17 \\ \hline 314.94564 \end{array}$$

medium:

medium: nam si pro m scribatur $99\frac{1}{2}$, habebitur Terminus in medio inter centesimum & centesimum primum. Vel si substituatur $99\frac{1}{3}$ pro m, habebitur Terminus consistens post centesimum tertiâ parte intervalli communis.

Notandum est, posse Terminorum alicujus Seriei reciprocos interpolari; sic reciproci Terminorum in Serie novissimâ constituunt Seriem $1, \frac{1}{2}A, \frac{3}{4}B, \frac{5}{6}C, \frac{7}{8}D$, &c. in qua Terminus ab initio remotus intervallo quovis æquali $\frac{1}{2}m$, æqualis erit Termino Seriei

$$1, \frac{a}{m+1}, \frac{3b}{m+3}, \frac{5c}{m+5}, \frac{7d}{m+7}, \text{\&c.}$$

qui consistit in medio inter primum & secundum; quique proinde invenietur esse media proportionalis inter Seriem sequentem

$$\frac{1}{m+1} + \frac{A}{2.\overline{m+3}} + \frac{9B}{4.\overline{m+5}} + \frac{25C}{6.\overline{m+7}} + \frac{49D}{8.\overline{m+9}} + \text{\&c.}$$

& numerum .6366197723676 qui æqualis est unitati divisæ per semicircumferentiam Circuli: id quod constabit insistendo vestigiis prioris partis hujus Exempli.

Exemplum II.

Quæratur Terminus hujus Seriei $1, \frac{2}{1}A, \frac{5}{4}B, \frac{8}{7}C, \frac{11}{10}D$, &c. quantumvis remotus ab initio; scilicet intervallo m: & per Propositionem decimam octavam Terminus ille æquabitur Termino hujus Seriei

$$1, \frac{2a}{3m+2}, \frac{5b}{3m+5}, \frac{8c}{3m+8}, \frac{11d}{3m+11}, \text{\&c.}$$

qui consistit ante primum tertia parte intervalli communis. Igitur intercalentur Numeratores & Denominatores seorsim ut in Exemplo secundo Propositionis vigesimæ primæ, scilicet per Logarithmos; & habebitur Terminus quæsitus.

Scholion.

Hinc constat Terminos Serierum longissime distantes erui posse non minus accurate ac intermedii sub initio. Sed in Serie interpolandâ 1,

$\frac{r}{p}$

$\frac{r}{p}$ A, $\frac{r+1}{p+1}$ B, &c. ſi differentia inter p & r ſit magna, magnus item erit labor in inveniendo quovis Termino. Caſus autem omnium facillimus eſt ubi $p-r$ eſt æqualis $\pm\frac{1}{2}$, ut in Exemplo primo, præterquam ubi differentia illa eſt numerus integer quo in Caſu Series erit accurate interpolabilis.

Propositio XXIII.

Invenire rationem quam habet Uncia media ad Summam omnium Unciarum in quacunque dignitate binomii.

Solutio prior.

Si index dignitatis ſit numerus par, appelletur n; vel ſi ſit impar, voceter $n-1$; eritque ut Uncia media ad Summam omnium ejuſdem dignitatis, ita Unitas ad mediam proportionalem inter ſemicircumferentiam Circuli & alterutram Serierum ſequentium

$$n+\frac{A}{2.\overline{n+2}}+\frac{9B}{4.\overline{n+4}}+\frac{25C}{6.\overline{n+6}}+\frac{49D}{8.\overline{n+8}}+\frac{81E}{10.\overline{n+10}}+\&c.$$

Vel

$$n+1-\frac{A}{2.\overline{n-3}}-\frac{9B}{4.\overline{n-5}}-\frac{25C}{6.\overline{n-7}}-\frac{49D}{3.\overline{n-9}}-\frac{81E}{10.\overline{n-11}}-\&c.$$

Exempli gratiâ, ſi quæratur ratio Unciæ mediæ ad Summam omnium in dignitate centeſimâ vel nonageſimâ nonâ, erit $n=100$; qui multiplicatus in ſemiperiferiam Circuli, producit primum Terminum $A=157.079632679$; dein erit $B=\frac{A}{204}$, $C=\frac{9B}{416}$, $D=\frac{25C}{636}$, &c. atque perficiendo computum ut in margine, invenietur Summa Terminorum 157.866984459, cujus Radix quadrata 12.5645129018 eſt ad unitatem ut Summa omnium Unciarum ad mediam in dignitate centeſimâ vel nonageſimâ nonâ. Eſt autem hic computus per priorem Seriem: nam licet exigua ſit differentia, præfero eam in quâ Termini ſunt omnes ejuſdem ſigni.

157.079632679
769998199
16658615
654820
37137
2734
246
26
3
157.866984459

Solutio

Solutio posterior.

Manente n ut antea; erit Summa omnium Unciarum ad mediam in subduplicatâ ratione periferiæ Circuli ad alterutram Serierum sequentium,

$$\frac{1}{n+1}+\frac{A}{2.\overline{n+3}}+\frac{9B}{4.\overline{n+5}}+\frac{25C}{6.\overline{n+7}}+\frac{49D}{8.\overline{n+9}}+\frac{81E}{10.\overline{n+11}}+\&c.$$

Vel

$$\frac{1}{n}-\frac{A}{2.\overline{n-2}}-\frac{9B}{4.\overline{n-4}}+\frac{25C}{6.\overline{n-6}}+\frac{49D}{8.\overline{n-8}}+\frac{81E}{10.\overline{n-10}}+\&c.$$

Vel quod eodem redit, ponatur $a = .6366197723676$, quoto scilicet qui prodit dividendo unitatem per semicircumferentiam Circuli; & media proportionalis inter numerum a & alterutram illarum Serierum erit ad unitatem ut Uncia media ad summam omnium.

Ut si sit index $n = 100$, ut in computo superiore; erit per Seriem primam $A = \frac{a}{101}$, $B = \frac{A}{206}$, $C = \frac{9B}{420}$, $D = \frac{25C}{642}$, &c. Ex calculo autem apposito patet Summam Terminorum esse .00633444670787, cujus Radix quadrata .0795892373872 est ad unitatem ut Uncia media ad Summam omnium in dignitate nonagesimâ nonâ vel centesimâ.

.00630316606305
3059789351
65566914
2553229
143473
10469
934
98
12
2
.00633444670787

Et sic omnino sunt quatuor Series ejusdem simplicitatis pro solutione hujus Problematis. Ceterum in praxi non opus est recurrere ad Series: nam sufficit sumere mediam proportionalem inter semicircumferentiam Circuli & $n+\frac{1}{2}$; hæc enim semper approximabit propius quam duo primi Termini Seriei, quorum etiam primus solus plerumque sufficit. Exempli gratiâ, si sit $n = 100$, erit $n+\frac{1}{2} = 100\frac{1}{2}$, qui ductus in semicircumferentiam Circuli producit 157.865, cujus Radix quadrata est 12.5644, unitate deficiens in ultimâ figurâ.

Eadem vero approximatio aliter & praxi forte commodior sic enunciatur. Pone c ad unitatem ut quadratum Diametri ad Circulum; hoc est, sit $c = 1.2732395447352$; eritque summa Unciarum ad mediam ut Uncia

Unitas ad $\sqrt{\frac{c}{2n+1}}$ quamproxime, existente errore in excessu circiter $\frac{1}{16nn}\sqrt{\frac{c}{2n+1}}$. Sit $n=100$, erit $\frac{c}{2n+1}=.006334525$, ejusque Radix quadrata .07958973 accurata est in sextâ decimali: quæ vero divisa per $16nn$, hoc est per 160000, dabit correctionem .00000050; & hæc subducta de Approximatione, relinquit numerum quæsitum .07958923 accuratum in ultima figurâ.

Similiter sit $n=900$, erit $\frac{c}{2n+1}=.000706962545$, cujus Radix quadrata .026588767 superat verum binario in ultimâ figurâ. Sin vero correctio computetur ac subducatur de Approximatione, habebitur numerus quæsitus accuratus in decimâ tertia decimali.

Hem vero Approximationem æque facilem & magis accuratam. Differentia inter Logarithmos numerorum $n+2$ & $n-2$, dividatur per 16, & quotus adjiciatur dimidio Logarithmi indicis n, huic dein summæ addatur Logarithmus constans .0980599385151, scilicet dimidium Logarithmi semicircumferentiæ Circuli, & summa novissima erit Logarithmus numeri qui est ad unitatem ut summa Unciarum ad mediam. Verbi gratiâ, sit $n=900$, computus erit

$\frac{1}{2}\ l,\ 900$	1.4771212547
16) Diff. Log. 902 & 898	.0001206376
Log. constans	.0980599385
Summa	1.5753018308

Et hæc Summa verum superat binario, in ultimâ figurâ, estque Logarithmus numeri 37.6098698 qui est ad unitatem ut summa Unciarum ad mediam in potestate 900 vel 899. Et si vis illius numeri reciprocum, sume complementum Logarithmi, scilicet —2.4246981692, & numerus eidem correspondens erit .0265887652 qui monstrat rationem Unciæ mediæ ad summam omnium in prædictis dignitatibus.

DEMONSTRATIO.

Dignitates Binomii, quarum indices sunt numeri pares, habent unam Unciam mediam; eæ vero, quarum indices sunt impares, habent duas Uncias medias. Et hinc oriuntur duo Casus Problematis. Primo quando index est par, divide summas Unciarum 1, 4, 16, 64, 256, 1024, &c. per suas Uncias medias, 1, 2, 6, 20, 70, 252, &c. & quoti 1, 2, $\frac{8}{3}$, $\frac{16}{5}$, $\frac{128}{35}$, $\frac{256}{63}$, &c. sive 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, $\frac{6}{5}$ C, $\frac{8}{7}$ D, &c. e-

 runt

runt ad unitatem ut fummæ Unciarum ad Uncias medias in diverfis dignitatibus.

Similiter fi fummæ Unciarum in dignitatibus imparibus, fcilicet 2, 8, 32, 128, 512, &c. dividantur per Uncias fuas medias 1, 3, 10, 35, 126, &c. rurfus prodibunt idem quoti, utpote 2, $\frac{8}{3}$, $\frac{16}{5}$, $\frac{128}{35}$, &c. Nam eadem eft relatio inter fummam Unciarum & Unciam mediam in quavis dignitate pare, quæ eft inter fummam Unciarum & Unciam mediam in dignitate impare proxime inferiori. Adeoque interpolatio Seriei, 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, $\frac{6}{5}$ C, $\frac{8}{7}$ D, &c. ut in Exemplo primo Propofitionis vigefimæ fecundæ, folvit utrumque Cafum Problematis. Sed hic dabimus inveftigationem harum Serierum abfque methodo Differentiali.

Analyfis Solutionis prioris.

Series interpolanda 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, &c. definitur Æquatione $T' = \frac{n+2}{n+1}T$, ubi n eft quantitas variabilis, ejufque valores fucceffivi 0, 2, 46, 8, &c. fcilicet indices dignitatum quando funt pares, vel indices aucti unitate quando funt impares. Quadra utramque partem Æquationis refolvendæ, & habebitur $T'T' = \frac{nn+4n+4}{nn+2n+1}TT$ vel quod idem eft $2T'T' + \overline{n+2} \times \overline{TT - T'T'} - \frac{T'T'}{n+2} = 0$. Affume jam

$$TT = An + \frac{Bn}{n+2} + \frac{Cn}{n+2.n+4} + \frac{Dn}{n+2.n+4.n+6} + \&c.$$

Et poft debitam reductionem juxta præcepta jam tradita, invenies

$$TT = An + B + \frac{C-2B}{n+2} + \frac{D-4C}{n+2.n+4} + \&c.$$

In quâ fcribe valores variabilium confequentes pro antecedentibus, hoc eft, $T'T'$ pro TT, & $n+2$ pro n, & emerget

$$T'T' = A\overline{n+2} + B + \frac{C-2B}{n+4} + \frac{D-4C}{n+4.n+6} + \&c.$$

Tum fumendo horum valorum differentiam, & ducendo eandem in $n+2$, prodibit

$$\overline{n+2} \times \overline{TT - T'T'} = -2A\overline{n+2} \quad * \quad + \frac{2C-4B}{n+4} + \frac{4D-16C}{n+4.n+6} + \&c.$$

Ceterum

Ceterum in valore ipsius TT prius assumpto, si scribatur $n+2$ pro n, habebitur

$$T'T' = \overline{n+2} \text{ in } A + \frac{B}{n+4} + \frac{C}{n+4.n+6} + \frac{D}{n+4.n+6.n+8} + \&c.$$

Atque dividendo per $n+2$,

$$\frac{T'T'}{n+2} = A + \frac{B}{n+4} + \frac{C}{n+4.n+6} + \frac{D}{n+4.n+6.n+8} + \&c.$$

Substitue jam in Æquatione resolvenda, valores ipsorum $T'T'$, $\overline{n+2} \times \overline{TT - T'T'}$ & $\frac{T'T'}{n+2}$, ad eandem formam reductos, & resultabit

$$2B - A + \frac{4C - 9B}{n+4} + \frac{6D - 25C}{n+4.n+6} + \frac{8E - 49D}{n+4.n+6.n+8} + \&c. = 0.$$

Denique ponendo Numeratores nihilo æquales, habebis $2B - A = 0$, $4C - 9B = 0$, $6D - 25C = 0$, $8E - 49D = 0$, &c. Et hæ sunt relationes Coefficientium in prima Serie. Et ad eundem modum eruitur Series posterior in Solutione priore.

Analysis Solutionis posterioris.

Solutio posterior perficitur interpolando Seriem 1, $\frac{1}{2}a$, $\frac{3}{4}b$, $\frac{5}{6}c$, $\frac{7}{8}d$, &c. cujus Termini sunt reciproci eorum in priore: ea vero definitur Æquatione $T' = \frac{n+1}{n+2}T$, in qua valores ipsius n successivi sunt 0, 2, 4, 6, 8, &c. ut ante. Et quadrando evadit $T'T' = \frac{nn+2n+1}{nn+4n+4}TT$, hoc est, $\overline{n+1.n+3}\ \overline{TT - T'T'} - 2.\overline{n+1}\ TT - T'T' = 0.$

Fingamus nunc

$$TT = \frac{A}{n+1} + \frac{B}{n+1.n+3} + \frac{C}{n+1.n+3.n+5} + \frac{D}{n+1.n+3.n+5\,n+7} + \&c.$$

Et substituendo $n+2$ pro n, proveniet

$$T'T' = \frac{A}{n+3} + \frac{B}{n+3.n+5} + \frac{C}{n+3.n+5.n+7} + \frac{D}{n+3.n+5.n+7.n+9} + \&c.$$

adeoque

adeoque

$$\overline{n+1}\cdot\overline{n+3}\cdot\overline{TT-TT} = 2A + \frac{4B}{n+5} + \frac{6C}{n+5.n+7} + \frac{8D}{n+5.n+7.n+9} + \&c.$$

quæ ad debitam formam reducta fit

$$2A + \frac{4B}{n+3} + \frac{6C-8B}{n+3.n+5} + \frac{8D-49C}{n+3.n+5.n+7} + \&c.$$

Scribe jam hosce valores in Æquatione resolvenda, atque resultabit

$$\frac{2B-A}{n+3} + \frac{4C-9B}{n+2.n+5} + \frac{6E-25C}{n+3.n+5.n+7} + \frac{8E-49D}{n+3.n+5.n+7.n+9} + \&c. = 0.$$

Ubi Numeratores æquati nihilo, dabunt relationem Coefficientium Seriei prioris in Solutione posteriore.

Quod autem Coefficiens A sit in uno Casu semicircumferentia Circuli, & ejusdem reciproca in altero, sic demonstratur. Per Seriem priorem est $TT = An$ in $1 + \frac{1}{2.n+2} + \&c.$ Et quo major est n, eo propius Æquatio $TT = An$, accedet ad veritatem, quoniam Termini posteriores tandem evadent infinite minores prioribus. Igitur in Æquatione $TT = An$, vel $A = \frac{TT}{n}$, si pro n successive scribantur ejus valores, 2, 4, 6, 8, 10, &c. & simul quadrata Terminorum correspondentium pro TT; orientur Æquationes continue approximantes ad verum ipsius A valorem, scilicet

$$A = 2,$$

$$A = 2 \times \frac{8}{9},$$

$$A = 2 \times \frac{8}{9} \times \frac{24}{25},$$

$$A = 2 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49},$$

&c.

Quare valor ipsius A est factum sub omnibus $2 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times \frac{80}{81} \times \&c.$ usque in infinitum; quod æquale est semicircumferentiæ Circuli per *Arithmeticam Infinitorum Wallisii.*

Propositio

7

Propositio XXIV.

Si in Ordinata Curvæ $x^{r+z-1}\times\overline{1-x}^{p-r-1}$ *scribantur successive numeri integri* 0, 1, 2, 3, 4, *&c. pro* z, *dico eandem esse relationem inter Areas Ordinatarum provenientium, quæ est inter Terminos Seriei* a, $\frac{r}{p}a$, $\frac{r+1}{p+1}b$, $\frac{r+2}{p+2}c$, $\frac{r+3}{p+3}d$, *&c. modo abscissa* x *sit æqualis unitati.*

Sint enim Areæ & Ordinatæ correspondentes

Areæ	Ordinatæ
A	$x^{r-1}\times\overline{1-x}^{p-r-1}$
B	$x^{r}\times\overline{1-x}^{p-r-1}$
C	$x^{r+1}\times\overline{1-x}^{p-r-1}$
D	$x^{r+2}\times\overline{1-x}^{p-r-1}$
E	$x^{r+3}\times\overline{1-x}^{p-r-1}$
&c.	&c.

Tum comparando hasce Areas per Propositionem septimam *Newtoni* de Quadraturâ Curvarum, invenies

$$B=\frac{rA-x^{r}\times\overline{1-x}^{p-r}}{p},$$

$$C=\frac{\overline{r+1}\,B-x^{r+1}\times\overline{1-x}^{p-r}}{p+1},$$

$$D=\frac{\overline{r+2}\,C-x^{r+2}\times\overline{1-x}^{p-r}}{p+2},$$

$$E=\frac{\overline{r+3}\,D-x^{r+3}\times\overline{1-x}^{p-r}}{p+3},$$

&c.

Sit

Sit jam x æqualis unitati, ut in Theoremate ponitur; eritque $1-x=0$; quo in caſu relatio Arearum fit $B=\frac{r}{p}A$, $C=\frac{r+1}{p+1}B$, $D=\frac{r+2}{p+2}C$, $E=\frac{r+3}{p+3}D$, &c. Adeoque eadem eſt relatio inter Areas harum Curvarum & Terminos Seriei propoſitæ, quando Abſciſſa x eſt unitas. Q. E. D.

Corollarium. Hinc in Serie a, $\frac{r}{p}a$, $\frac{r+1}{p+1}b$, $\frac{r+2}{p+2}c$, &c. ſi z denotet intervallum inter primum Terminum a, & alium quemvis T primarium vel intermedium; erit ut Area Curvæ cujus Ordinata eſt $x^{r-1}\times\overline{1-x}^{p-r-1}$ ad Aream Curvæ cujus Ordinata eſt $x^{r+z-1}\times\overline{1-x}^{p-r-1}$, ita Terminus primus a ad alium quemvis primarium vel intermedium, intervallo z diſtantem ab initio.

Exemplum I.

Detur Series intercalanda 1, $\frac{1}{2}a$, $\frac{3}{4}b$, $\frac{5}{6}c$, $\frac{7}{8}d$, &c. Quoniam differentia communis tum Numeratorum tum Denominatorum eſt 2; divide eoſdem per binarium, ut differentia illa fiat unitas ut in Theoremate; & evadet Series 1, $\frac{\frac{1}{2}}{1}a$, $\frac{\frac{3}{2}}{2}b$, &c. quæ collata cum ea in Propoſitione, dat $p=1$, $r=\frac{1}{2}$; quibus ſubſtitutis, erit ut Area Ordinatæ $x^{-\frac{1}{2}}\times\overline{1-x}^{-\frac{1}{2}}$ ad Aream Ordinatæ $x^{z-\frac{1}{2}}\times\overline{1-x}^{-\frac{1}{2}}$, id eſt, ut Area hujus $\frac{1}{\sqrt{x-xx}}$ ad Aream hujus $\frac{x^z}{\sqrt{x-xx}}$ ita primus Terminus Seriei, ſive unitas ad alium quemvis primarium vel intermedium intervallo z diſtantem ab initio.

Ut ſi Terminus deſideratus conſiſtat in medio inter primum & ſecundum, erit $z=\frac{1}{2}$, quo in Caſu fit Ordinata poſterior $\frac{x^{\frac{1}{2}}}{\sqrt{x-xx}}$, ſive $\frac{1}{\sqrt{1-x}}$: adeoque ut Area Ordinatæ $\frac{1}{\sqrt{x-xx}}$ ad Aream Ordinatæ $\frac{1}{\sqrt{1-x}}$, id eſt, ut Circumferentia Circuli 3.1415926 - - - &c. ad 2, ita unitas ad Terminum in medio inter primum & ſecundum; qui hoc pacto prodit .63661977 - - - &c.

Si

Si ejusdem Seriei quæratur Terminus centesimus primus, erit $z = 100$; adeoque ut 3.1415 - - - &c. ad Aream Ordinatæ $\frac{x^{100}}{\sqrt{x-xx}}$, ita unitas ad Terminum Propositum. Et similiter ponendo $z = 100\frac{1}{2}$, Terminus in medio inter centesimum primum & centesimum secundum determinabitur per Circumferentiam Circuli & Aream Ordinatæ $\frac{x^{100}}{\sqrt{1-x}}$: ubique tamen sumendo partes Arearum quæ jacent supra Abscissam unitati æqualem.

Possunt etiam Terminorum reciproci interpolari, idque nonnunquam magis commode quam ipsi Termini. Reciproci Terminorum in novissima Serie sunt 1, $\frac{2}{1}a$, $\frac{4}{3}b$, &c. adeoque est $r = 1$, $p = \frac{1}{2}$; & inde ut Area Ordinatæ $x^0 \times \overline{1-x}^{-\frac{3}{2}}$ ad Aream Ordinatæ $x^z \times \overline{1-x}^{-\frac{3}{2}}$ ita primus Terminus ad illum distantem intervallo z ab initio: id est, ut 2 ad Aream Ordinatæ $\frac{x^z}{1-x\sqrt{1-x}}$; nam prior Curvæ est quadrabilis.

Exemplum II.

Interpolanda sit Series 1, $\frac{1}{2}a$, $\frac{4}{5}b$, $\frac{7}{8}c$, $\frac{10}{11}d$, &c. Divide Numeratores & Denominatores per eorum incrementum 3; & invenies $p = \frac{2}{3}$, $r = \frac{1}{3}$; hinc fit Ordinata prior $x^{-\frac{2}{3}} \times \overline{1-x}^{-\frac{2}{3}}$, sive $\frac{1}{\sqrt[3]{x^2 - 2x^3 + x^4}}$; & posterior $\frac{x^z}{\sqrt[3]{x^2 - 2x^3 + x^4}}$. Dein ut Area illius ad Aream hujus ita primus Terminus Seriei ad alium quemcunque cujus distantia ab initio est z.

Propositio XXV.

Si in Ordinata Curvæ $x^{p-z} \times \overline{1-x}^{r+z-1}$ scribantur successive numeri integri 0, 1, 2, 3, 4, &c. pro z, eadem erit relatio inter Areas Ordinatarum provenientium quæ

quæ eſt inter Terminos Seriei a, $\frac{r}{p}a$, $\frac{r+1}{p-1}b$, $\frac{r+2}{p-2}c$, $\frac{r+3}{p-3}d$, *&c. Ubi Numeratores continue increſcunt, Denominatores vero decreſcunt. Et hic quoque pono Abſciſſam* x *unitati æqualem.*

Hæc Propoſitio demonſtratur ad modum ſuperioris.

Corollarium. Hinc in Serie a, $\frac{r}{p}a$, $\frac{r+1}{p-1}b$, $\frac{r+2}{p-2}c$, $\frac{r+3}{p-3}d$, &c. ut Terminus primus a ad alium quemvis intervallo z diſtantem a principio, ita Area Curvæ cujus Ordinata eſt $x^p \times \overline{1-x}^{r-1}$ ad Aream Curvæ cujus Ordinata eſt $x^{p-z} \times \overline{1-x}^{r+z-1}$.

Exemplum I.

Detur Series interpolanda 1, $\frac{n}{1}a$, $\frac{n-1}{2}b$, $\frac{n-2}{3}c$, $\frac{n-3}{4}d$, &c. cujus Termini ſunt Unciæ binomii in dignitate cujus index eſt n. Quoniam hæc Series non cadit directe ſub hâc Propoſitione, interpolo Terminorum reciprocos 1, $\frac{1}{n}a$, $\frac{2}{n-1}b$, $\frac{3}{n-2}c$, &c. quo pacto erit $r=1$, $p=n$; adeoque ut Area Ordinatæ $x^n \times \overline{1-x}^0$ ad Aream Ordinatæ $x^{n-z} \times \overline{1-x}^z$, ita unitas ad Terminum intervallo z diſtantem a principio, in Serie poſteriore: vel ut Area Ordinatæ $x^{n-z} \times \overline{1-x}^z$ ad $\frac{1}{n+1}$, ita unitas ad Terminum in priore Serie, intervallo z remotum ab initio.

Ut ſi deſideretur Uncia Termini quinti in dignitate nona, erit $n=9$, $z=4$; quibus ſcriptis, erit Area Ordinatæ $x^5 \times \overline{1-x}^4$ ad $\frac{1}{10}$ ut unitas ad Unciam quæſitam. Ordinata autem evoluta eſt $x^5 - 4x^6 + 6x^7 - 4x^8 + x^9$, ejuſque Area $\frac{1}{6} - \frac{4}{7} + \frac{6}{8} - \frac{4}{9} + \frac{1}{10}$, ſive $\frac{1}{1260}$; Tum ut $\frac{1}{1260}$ ad $\frac{1}{10}$ ita unitas ad 126, quæ eſt Uncia propoſita.

Exemplum

Exemplum II.

Si in poteſtate ſimplici Binomii, quæratur Terminus qui conſiſtit in medio inter duas Uncias 1 & 1; erit index binomii $n=1$, $z=\frac{1}{2}$; & exinde ut Area Ordinatæ $x^{\frac{1}{2}} \times \overline{1-x}^{\frac{1}{2}}$ ad $\frac{1}{2}$, hoc eſt, ut Area Circuli ad quadratum Circumſcriptum ita unitas ad Terminum inter Uncias 1 & 1.

Scholion.

Quando Curvæ quadrandæ ſunt plurimarum dimenſionum, inveniendæ ſunt aliquot earum Ordinatæ per Tabulam Logarithmorum; ex quibus dabuntur Areæ per Parabolam *Newtoni*. Quod ſi relatio Terminorum in Serie interpolanda ſit inter plures Terminos, abſolvetur interpolatio per Comparationem aliarum Curvarum. Sed hanc materiam miſſam faciens adjungam quædam de aliis methodis interpolandi.

Propositio XXVI.

Si Series interpolanda ſit $1,\ \frac{r}{p}a,\ \frac{r+1}{p+1}b,\ \frac{r+2}{p+2}c,\ \frac{r+3}{p+3}d,$ *&c. & ponatur* $n=r-p$, *atque*

$$B=\frac{n}{1}.\frac{n-1}{2}\ in\ A,$$

$$C=\frac{n-1}{2}.\frac{n-2}{2}\ in\ B+\frac{n}{3}A,$$

$$D=\frac{n-2}{3}.\frac{n-3}{2}\ in\ C+\frac{n-1}{3}B+\frac{n}{3}.\frac{n-1}{4}A,$$

$$E=\frac{n-3}{4}.\frac{n-4}{2}\ in\ D+\frac{n-2}{3}C+\frac{n-1}{3}.\frac{n-2}{4}B+\frac{n}{3}.\frac{n-1}{4}.\frac{n-2}{5}A,$$

$$F=\frac{n-4}{5}.\frac{n-5}{2}\ in\ E+\frac{n-3}{3}D+\frac{n-2}{3}.\frac{n-3}{4}C+\frac{n-1}{3}.\frac{n-2}{4}.\frac{n-3}{5}B$$
$$+\frac{n}{3}.\frac{n-1}{4}.\frac{n-2}{5}.\frac{n-3}{6}A,$$

&c.

 Tum

Tum erit z^n *in* $A+\frac{B}{z}+\frac{C}{z^2}+\frac{D}{z^3}+\frac{E}{z^4}+$ *&c. Terminus Seriei interpolandæ primarius vel intermedius, cujus distantia ab initio est* $z-p$.

Notandum est Coefficientem A, per Propositionem decimam octavam æqualem esse Termino in Serie Numeratorum 1, ra, $\overline{r+1}.b$, $\overline{r+2}.c$, &c. distantem intervallo $p-r$ ab initio; eamque determinari per Exemplum secundum Propositionis vigesimæ primæ.

DEMONSTRATIO.

Series proposita definitur Æquatione differentiali $T'=\frac{z+n}{z}T$, ubi est $n=r-p$ ut in Theoremate & successivi valores indeterminatæ z sunt p, $p+1$, $p+2$, &c. Finge jam

$$T=z^n \text{ in } A+\frac{B}{z}+\frac{C}{z^2}+\frac{D}{z^3}+\frac{E}{z^4}+\&c.$$

dein pro T & z scribe earum valores succedentes T' & $z+1$, respective, & emerget

$$T'=\overline{z+1}^n \text{ in } A+\frac{B}{z+1}+\frac{C}{\overline{z+1}^2}+\frac{D}{\overline{z+1}^3}+\frac{E}{\overline{z+1}^4}+\&c.$$

Vel evolvendo dignitates

$$T'=z^n \text{ in } A+ \frac{nA+B}{z}+\frac{\overline{nA+2B}.\overline{n-1}+2C}{z^2}+\frac{\overline{nA+3B}.\overline{n-1}.\overline{n-2}+6.\overline{n-1}.C+6D}{z^3}+\&c.$$

Æquatio resolvenda $T'=\frac{z+n}{z}T$, hunc in modum scribatur $T'z-Tz-Tn=0$; & in eadem substituantur valores ipsorum T & T', atque resultabit

$$z^n \text{ in } \frac{n.\overline{n-1}A-2B}{2z}+\frac{\overline{nA+3B}.\overline{n-1}.\overline{n-2}-12C}{6z^2}+\&c.=0.$$

Ponantur jam Numeratores æquales nihil, & habebuntur $B=\frac{n}{1}.\frac{n-1}{2}$ in A, $C=\frac{n-1}{2}.\frac{n-2}{2}$ in $B+\frac{n}{3}A$, &c. Atque producendo computum, reliquæ Coefficientes prodibunt in Theoremate. Q. E. D.

EXEMPLUM

Exemplum I.

Proponatur Series 1, $\frac{1}{2}a$, $\frac{3}{4}b$, $\frac{5}{6}c$, $\frac{7}{8}d$, &c. quæ definitur Æquatione $T' = \frac{z - \frac{1}{2}}{z} T$, in qua valores succeſſivi ipſius z ſunt 1, 2, 3, 4, &c. hæc comparata cum Æquatione $T' = \frac{z-n}{z} T$, dat $n = -\frac{1}{2}$; & exinde

$$B = \frac{3}{8} \text{ in } A \qquad = A \times \frac{3}{8},$$

$$C = \frac{15}{16} \text{ in } B - \frac{1}{6} A \qquad = A \times \frac{25}{128},$$

$$D = \frac{35}{24} \text{ in } C - \frac{3}{6} B + \frac{1}{16} A \qquad = A \times \frac{105}{1024},$$

$$E = \frac{63}{32} \text{ in } D - \frac{5}{6} C + \frac{5}{16} B - \frac{1}{32} A \qquad = A \times \frac{1659}{32768},$$

$$F = \frac{99}{40} \text{ in } E - \frac{7}{6} D + \frac{35}{48} C - \frac{7}{32} B + \frac{7}{384} A \qquad = A \times \frac{6237}{262144},$$

&c.

Adeoque

$$T = \frac{A}{\sqrt{z}} \text{ in } 1 + \frac{3}{8z} + \frac{25}{128z^2} + \frac{105}{1024z^3} + \frac{1659}{32768z^4} + \frac{6237}{262144z^5} + \&c.$$

Quantitas autem A in hujuſmodi Exemplis ſic determinari poſſit. Per relationem Terminorum interpolandorum quære primarium quemvis ſatis longe diſtantem ab initio, verbi gratiâ decimum ſextum, qui hic prodit .144464448 --- &c. Hunc ſcribe pro T & interea pro z valorem ſuum correſpondentem, nempe 16; atque habebis

$$.144464448 = \frac{A}{4} \text{ in } 1 + \frac{3}{8.16} + \frac{25}{128.16.16} + \&c.$$

vel colligendo Terminos in unam Summam, $.144464448 = \frac{A}{4}$ in 1.02422627, unde prodit $A = .564189583548$: quo dato dabitur T in alio quovis caſu per Terminos pauciſſimos ſui valoris.

Exemplum

EXEMPLUM II.

Interpolanda sit Series 1, $\frac{2}{1}a$, $\frac{5}{4}B$, $\frac{8}{7}c$, $\frac{11}{10}d$, &c. quæ definitur Æquatione $T' = \frac{z+\frac{1}{3}}{z}T$, existentibus $\frac{1}{3}$, $\frac{4}{3}$, $\frac{7}{3}$, $\frac{10}{3}$, &c. valoribus Abscissæ z successivis: hæc comparata cum Æquatione in Theoremate dat $n = \frac{1}{3}$; quo substituto obtinebitur

$$T = A\sqrt[3]{z} \text{ in } 1 - \frac{1}{9z} \; * \; + \frac{5}{2187z^3} + \frac{253}{19683z^4} + \frac{5610}{177147z^5} + \&c.$$

Jam ad eruendum Coefficientem A, quæro Terminum decimum quartum Seriei interpolandæ, qui prodit 4.652136: hunc deinde valorem scribo pro T, & pro z valorem suum decimum quartum $\frac{40}{3}$, atque habeo

$$4.652136 = A\sqrt[3]{\frac{40}{3}} \text{ in } 1 - \frac{1}{120} + \frac{1}{345600} + \&c.$$

Vel extrahendo radicem cubicam numeri $\frac{40}{3}$, & colligendo Terminos in unam summam, obtineo $4.652136 = A \times 2.351505$, adeoque $A = 1.978365$. Datâ nunc A, Terminus quilibet alius invenietur summa facilitate. Quæratur is qui consistit tertiâ parte intervalli communis ante millesimum primum: pro z scribe suum valorem correspondentem 1000, & valor ipsius T prodibit 10 A in $1 - \frac{1}{9000}$, sive $T = 19.78343$. Nam ubi Terminus desideratus longe distat a principio & computus non est producendus ad plurimas figuras perpauci Termini in valore ipsius T abunde sufficiunt.

SCHOLION.

Eodem modo quo in hâc Propositione Radix extrahitur ex Æquatione $T' = \frac{z+n}{z}T$, extrahitur etiam ex aliâ quâvis quæ sub hâc formâ continetur

$$T \times \overline{z^\theta + az^{\theta-1} + bz^{\theta-2} + \&c.} = T' \times \overline{z^\theta + cz^{\theta-1} + dz^{\theta-2} + \&c.}$$

Nam

Nam eſt index $n=a-c$ per Propoſitionem ſextam; & inde forma Seriei pro T aſſumenda erit

$$T = Az^{n} + Bz^{n-1} + Cz^{n-2} + \&c.$$

Quippe in hujuſmodi Seriebus quæ ſunt Radices Æquationum Differentialium, indices ipſius z habent Unitatem pro decremento, prætérquam in quibuſdam caſibus valde particularibus. Habito igitur indice ipſius z in primo Termino habetur forma Seriei aſſumendæ pro Radice T: dein ſcribendo $z+1$ pro z, & T' pro T, prodibit

$$T' = A\overline{z+1}^{n} + B\,\overline{z+1}^{n-1} + C\,\overline{z+1}^{n-2} + \&c.$$

Tum reducendo hunc valorem ad formam ipſius T, ut ſupra oſtenſum eſt, & ducendo utrumque valorem in quantitates quas præcipit Æquatio reſolvenda; dabuntur Coefficientes aſſumptæ ex collatione membrorum homologorum in Æquatione reſultante

Propositio XXVII.

Si Æquatio ad Seriem ſit $T' = \frac{zz}{zz+r}T$, *erit Radix*

$$T = A + \frac{r}{z}A + \frac{r+1}{2.\overline{z+2}}B + \frac{r+4}{3.\overline{z+3}}C + \frac{r+9}{4.\overline{z+4}}D + \frac{r+16}{5.\overline{z+5}}E + \&c.$$

Æquatio reſolvenda $T' = \frac{zz}{zz+r}T$ hunc in modum ſcribatur $zzT - zzT' - rT' = 0$, & aſſumatur

$$T = A + \frac{B}{z} + \frac{C}{z.\overline{z+1}} + \frac{D}{z.\overline{z+1}.\overline{z+2}} + \frac{E}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Eritque

$$T' = A + \frac{B}{z+1} + \frac{C}{\overline{z+1}.\overline{z+2}} + \frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{E}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Unde

$$T - T' = \frac{B}{z.\overline{z+1}} + \frac{2C}{z.\overline{z+1}.\overline{z+2}} + \frac{3D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{4E}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Et ducendo in zz

$$zzT - zzT' = \frac{Bz}{z+1} + \frac{2Cz}{\overline{z+1}.\overline{z+2}} + \frac{3Dz}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{4Ez}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Atque per reductionem

$$zzT - zzT' = B + \frac{2C-B}{z+1} + \frac{3D-4C}{\overline{z+1}.\overline{z+2}} + \frac{4E-9D}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Hunc

Hunc valorem ſcribe pro $zzT - zzT'$, & pro T' ſuum valorem hactenus repertum, atque reſultabit

$$B - rA + \frac{2C - \overline{r+1}B}{z+1} + \frac{3D - \overline{r+4}C}{z+1.z+2} + \frac{4E - \overline{r+9}D}{z+1.z+2.z+3} + \&c. = 0.$$

Et per collationem membrorum homologorum, $B = \frac{r}{1}A$, $C = \frac{r+1}{2}B$, $D = \frac{r+4}{3}C$, $E = \frac{r+9}{4}D$, &c. Hi ſunt valores Coefficientium: quod ſi A, B, C, &c. denotent totos Terminos, prodibit valor ipſius T jam aſſignatus. Q. E. D.

Exemplum.

Invenit *Walliſius*, ultimum Terminum hujus Seriei $1, \frac{8}{9}A, \frac{24}{25}B, \frac{48}{49}C, \frac{80}{81}D$, &c. eſſe Aream Circuli cujus Diameter eſt unitas; ubi Denominatores ſunt quadrata numerorum imparium, & unitate majores Numeratoribus. Videamus autem hic quiſnam ſit ultimus Terminus hujus $1, \frac{4}{3}A, \frac{16}{15}B, \frac{36}{35}C, \frac{64}{63}D$, &c. ubi Numeratores ſunt quadrata numerorum parium & unitate majores Denominatoribus. Æquatio ad hanc Seriem erit $T' = \frac{zz}{zz - \frac{1}{4}}T$, exiſtentibus 1, 2, 3, 4, &c. valoribus Abſciſſæ z ſucceſſivis: ergo comparando hanc Æquationem cum illâ in Propoſitione, erit $r = -\frac{1}{4}$, quo ſubſtituto prodit

$$T = A - \frac{A}{4z} + \frac{3B}{8.\overline{z+1}} + \frac{15C}{12.\overline{z+2}} + \frac{35D}{16.\overline{z+3}} + \frac{63E}{20.\overline{z+4}} + \&c.$$

Et ad determinandum Coefficientem A, quære Terminum Seriei decimum, nempe 1.5300.1727.35, quem ſubſtitue pro T, & interea pro z valorem ſuum correſpondentem 10; atque obtinebis

$$1.5300.1727.35 = A \text{ in } 1 - \frac{1}{4.10} - \frac{1.3}{4.8.10.11} - \frac{1.3.15}{4.8.12.10.11.12} - \&c.$$

Hoc eſt, colligendo Terminos in unum, 1.5300.1727.35 = A in .9740.392454, & exinde A = 1.57079633, utique ſemicircumferentiæ Circuli: quo dato, facillime dabitur Terminus quilibet Seriei interpolandæ primarius vel intermedius. Conſtat autem ultimum ejus Terminum,

minum, five factum fub omnibus $\frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \frac{100}{99} \times$ &c. æquale effe primo Coefficienti A, adeoque femiperiferiæ Circuli.

Propositio XXVIII.

Invenire fummam quotcunque Logarithmorum, quorum numeri funt in progreffione Arithmetica.

Defignent $x+n$, $x+3n$, $x+5n$, $x+7n$, --- $z-n$, quotcunque numeros in progreffione Arithmeticâ, quorum primus & $x+n$, ultimus $z-n$, & communis differentia $2n$. Infuper denotent l,z & l,x Logarithmos Tabulares numerorum z & x; fitque $a = .43429.44819.03252$, fcilicet reciproco Logarithmi naturalis Denarii. Atque fumma Logarithmorum propofitorum æqualis erit differentiæ inter Series duas fequentes

$$\frac{zl,z}{2n} - \frac{az}{2n} - \frac{an}{12z} + \frac{7an^3}{360z^3} - \frac{31an^5}{1260z^5} + \frac{127an^7}{1680z^7} - \frac{511an^9}{1188z^9} + \&c.$$

$$\frac{xl,x}{2n} - \frac{ax}{2n} - \frac{an}{12x} + \frac{7an^3}{360x^3} - \frac{31an^5}{1260x^5} + \frac{127an^7}{1680x^7} - \frac{511an^9}{1188x^9} + \&c.$$

Hæ autem Series fic continuantur in infinitum; pone

$$-\frac{1}{3.4} = A,$$

$$-\frac{1}{5.8} = A + 3B,$$

$$-\frac{1}{7.12} = A + 10B + 5C,$$

$$-\frac{1}{9.16} = A + 21B + 35C + 7D,$$

$$-\frac{1}{11.20} = A + 36B + 126C + 84D + 9F,$$

&c.

Ubi numeri qui multiplicantur in A, B, C, D, &c. in diverfis valoribus funt Unciæ alternæ in dignitatibus imparibus binomii. Hifce præmiffis, erit Coefficiens Termini tertii $-\frac{1}{12} = A$, is quarti $+\frac{7}{360} = B$, quinti $-\frac{31}{1260} = C$, & fic deinceps.

Demon-

DEMONSTRATIO.

Minuatur variabilis z decremento ſuo $2n$; vel quod idem eſt, ſubſtituatur $z-2n$ pro z in Serie

$$\frac{zl,z}{2n}-\frac{az}{2n}-\frac{an}{12z}+\frac{7an^3}{360z^3}-\frac{31an^5}{1260z^5}+\&c.$$

& proveniet valor ejuſdem ſucceſſivus

$$\frac{\overline{z-2n}l,\overline{z-2n}}{2n}-\frac{a}{2n}\times\overline{z-2n}-\frac{an}{12.\overline{z-2n}}+\frac{7an^3}{360.\overline{z-2n}^3}-\frac{31an^5}{1260.\overline{z-2n}^5}+\&c.$$

Hunc ſubducito de valore priore, Terminis prius ad eandem formam per Diviſionem reductis, & relinquetur $lz-\frac{an}{z}-\frac{an^2}{2z^2}-\frac{an^3}{3z^3}-\frac{an^4}{4z^4}-$ &c. id eſt, Logarithmus numeri $z-n$. Adeoque univerſaliter decrementum duorum valorum ſucceſſivorum Seriei, æquatur Logarithmo ipſius $z-n$; qui exprimit in genere quemvis Logarithmorum qui erant ſummandi. Igitur Series erit ſumma Logarithmorum propoſitorum, ſi ab eâdem ſubducatur altera Series. Nam ſummæ perinde ac Areæ nonnunquam corrigendæ ſunt, ut evadant veræ.

EXEMPLUM I.

Proponatur invenire ſummam Logarithmorum decem numerorum 101, 103, 105, 107, 109, 111, 113, 115, 117, 119; hi collati cum hiſce $x+n$, $x+3n$, $x+5n$, - - - - $z-n$, dant differentiam communem $2n=2$, & $n=1$; atque primum $x+1=101$, ſecundum vero $z-1=119$: unde $x=100$, $z=120$. Hiſce autem ſubſtitutis, & .43429.44819.03252 pro a; atque Logarithmis ipſorum 100 & 120 reſpective pro l,x & l,z; valores duarum Serierum invenientur 78.28491.40012.1 & 98.69290.42601.6, quorum differentia dat 20.40799.02589.5 pro Summa Logarithmorum deſiderata.

EXEMPLUM II.

Quæratur jam Summa Logarithmorum numerorum 11, 12, 13, - - - 1000, quorum primus eſt 11, & ultimus 1000, & differentia communis unitas. Eſt igitur $n=\frac{1}{2}$, $x+\frac{1}{2}=10$, $z-\frac{1}{2}=1000$; unde $x=\frac{21}{2}$, $z=\frac{2001}{2}$; quibus ſcriptis, & Logarithmis ipſorum $\frac{21}{2}$ & $\frac{2001}{2}$ pro l,x & l,z; prodeunt pro valoribus Serierum 2567.20555.42879 &

6.16067.30987,

6.16067.30987, quorum differentia relinquit 2561.04488.11892 pro Summâ Logarithmorum quæsitâ.

Ceterum si velis summam quotcunque Logarithmorum numerorum naturalium 1, 2, 3, 4, 5, &c. pone $z-n$ esse ultimum numerorum, existente $r=\frac{1}{2}$; & tres vel quatuor Termini hujus Seriei $zl, z-az-\frac{a}{24z}+\frac{7a}{2880z^3}-$ &c. additi Logarithmo circumferentiæ Circuli cujus Radius est Unitas, id est, huic 0.39908.99341.79 dabunt summam quæsitam, idque eo minore labore quo plures Logarithmi sunt summandi. Sic si ponas $z-\frac{1}{2}=1000$, vel $z=\frac{2001}{2}$ valor Seriei erit 2567.20555.42879 ut antea; qui adjectus Logarithmo constanti efficit 2567.60464.42221 pro summâ Logarithmorum, primorum mille numerorum hujus Seriei 1, 2, 3, 4, 5, &c

EXEMPLUM III.

Oporteat invenire Unciam quingentesimam in dignitate Binomii millesimâ. Constat per Theorema *Newtoni* pro evolvendo Binomio Unciam illam æqualem esse producto sub quadringenti & nonaginta novem factoribus $\frac{1000}{1}, \frac{999}{2}, \frac{998}{3}, \frac{997}{4}, \frac{996}{5}, \text{- - - -} \frac{502}{499}$; quorum primus est $\frac{1000}{1}$, & ultimus $\frac{502}{499}$; existentibus tum Numeratoribus tum Denominatoribus in progressione Arithmetica. Ad inveniendam summam Logarithmorum Numeratorum 1000, 999, 998, 997, - - - - 502; pone differentiam communem $1=2n$, maximum eorum $1000=z-\frac{1}{2}$, minimum $502=x+\frac{1}{2}$; eritque $n=\frac{1}{2}$, $z=1000\frac{1}{2}$, $x=501$: quibus substitutis, prodibit 2567.20555.42879 pro valore prioris Seriei, & 1136.38715.63268 pro valore posterioris: horum autem differentia 1430.81839.79611 æqualis est summæ Logarithmorum Numeratorum. Dein ut obtineatur summa Logarithmorum Denominatorum 1, 2, 3, 4, - - - - 499; pone $n=\frac{1}{2}$, $z-\frac{1}{2}=499$, sive $z=499\frac{1}{2}$; & hisce scriptis in Serie priore, prodibit ejus valor 1130.98834.85966, cui adjiciatur Logarithmus .39908.99342 juxta regulam in Exemplo priore traditam: & habebis 1131.38743.85308 pro summa Logarithmorum Denominatorum. Quam denique subducito de summa Logarithmorum Numeratorum, atque manebit 299.43095.94303 Logarithmus scilicet Unciæ desideratæ.

SCHOLION.

Hujufmodi Series $1, \frac{r}{p}A, \frac{r+1}{p+1}B, \frac{r+2}{p+2}C$, &c. interpolatur per Propofitionem vigefimam fextam, ubi differentia inter r & p eft exigua: & per hanc Propofitionem generaliter nullâ ratione habitâ iftius differentiæ. Atque eodem plane modo invenire licet fummam Logarithmorum Numerorum, qui funt longe magis compofiti quam æquidifferentes; eoque pacto affignare Terminos Serierum quarum intercalatio pro difficillima haberi folet. Per hoc quoque Problema inveniuntur Areæ Curvarum quarum Ordinatæ funt hujufmodi $\overline{1-x}|^{1000}$, ubi index Binomii eft permagnus; fed in eo cafu folo quando pars Areæ quæfita jacet fupra partem Abfciffæ æqualem Unitati.

Et fane omnia fere Problemata de Interpolationibus huic Analyfi fubjiciuntur, imo etiamfi tres vel plures Termini Seriei intercalandæ ingrediantur Æquationem Differentialem: harum enim refolutionem in poteftate habeo. Atque hic notare libet Series quæ prodeunt per Parabolam *Newtoni*, prodire etiam per noftram methodum. Proponatur enim intercalatio Seriei

$$a, \frac{r+n}{r}a, \frac{r+n+1}{r+1}b, \frac{r+n+2}{r+2}c, \frac{r+n+3}{r+3}d, \&c.$$

quæ definitur Æquatione $T' = \frac{z+n+r}{z+r}T$, in qua valores Abfciffæ z fucceffivi funt 0, 1, 2, 3, &c. Ea vero fcribatur in modum fequentem $\overline{z+r} \times \overline{T'-T} - nT = 0$; & fingatur

$$T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Atque fubftituendo T' pro T, & $z+1$ pro z, obtinebitur

$$T' = A + B\overline{z+1}. + C\overline{z+1}.z + D\overline{z+1}.z.\overline{z-1} + E\overline{z+1}.z.\overline{z-1}.\overline{z-2} + \&c.$$

Unde $T' - T = B + 2Cz + 3Dz.\overline{z-1} + 4Ez.\overline{z-1}.\overline{z-2} + \&c.$

Hifce autem valoribus fubftitutis in Æquatione refolvenda, & membris reductis ad eandem formam, refultabit

$$\left.\begin{matrix}+rB\\-nA\end{matrix}\right\} \left.\begin{matrix}+2.\overline{r+1}C\\-\overline{n-1}B\end{matrix}\right\}z \left.\begin{matrix}+3.\overline{r+2}D\\-\overline{n-2}C\end{matrix}\right\}z.\overline{z-1} \left.\begin{matrix}+4.\overline{r+3}E\\-\overline{n-3}D\end{matrix}\right\}z.\overline{z-1}.\overline{z-2} + \&c. = 0.$$

Denique ponendo membra homologa æqualia nihil, habebitur $B = \frac{n}{r}A$, $C = \frac{1}{2} \times \frac{n-1}{r+1}B$, $D = \frac{1}{3} \times \frac{n-2}{r+2}C$, $E = \frac{1}{4} \times \frac{n-3}{r+3}D$, &c.

Adeoque

T =

$$T = A + A \times \frac{n}{r} \times \frac{z}{1} + A \times \frac{n}{r} \times \frac{z}{1} \times \frac{n-1}{r+1} \times \frac{z-1}{r+2} + \&c.$$

Hoc eſt

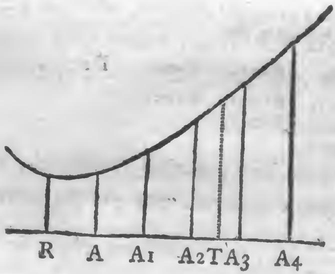
$$T = A + \frac{n}{r} A \frac{z}{1} + \frac{n-1}{r+1} B \frac{z-1}{2} + \frac{n-2}{r+2} C \frac{z-2}{3} + \frac{n-3}{r+3} D \frac{z-3}{4} \&c.$$

Ubi A, B, C, D, &c. non amplius denotant Coefficientes, ſed Terminos totos. Et prima Coefficiens A quæ ex Æquatione non determinatur, æqualis eſt Termino Seriei interpolandæ qui tranſit per initium Abſciſſæ z. Et hic eſt ipſiſſimus valor Termini T qui prodiiſſet per Propoſitionem decimam nonam. Atque etiam Series quas exhibet Propoſitio vigeſima invenientur per reſolutionem Æquationum Differentialium, aſſumendo debitam formam radicis.

Propositio XXIX.

Detur Series Ordinatarum intervallis quibuſcunque ab invicem diſtantium, pergens vero ex una tantum parte in infinitum; & oporteat invenire Lineam Parabolicam quæ tranſit per extremitates omnium.

Sint A, A1, A2, A3, A4, &c. Ordinatæ inſiſtentes Abſciſſæ in angulis rectis; ſitque R punctum quodlibet in Abſciſſâ; & ponatur $a =$ RA, $b =$ RA1, C $=$ RA2, $d =$ RA3, $e =$ RA4, &c. ſcilicet ſint a, b, c, d, e, &c. intervalla inter Ordinatas & punctum R reſpective. Atque deſignet T Ordinatam quamvis in genere cujus diſtantia a puncto R ſit z. Dein ponatur



B =

$$B = \frac{A1 - A}{b - a},\quad C = \frac{B1 - B}{c - a},\quad D = \frac{C1 - C}{d - a},\quad E = \frac{D1 - D}{e - a},\ \&c.$$

$$B1 = \frac{A2 - A1}{c - b},\quad C1 = \frac{B2 - B1}{d - b},\quad D1 = \frac{C2 - C1}{e - a},\ \&c.$$

$$B2 = \frac{A3 - A2}{d - c},\quad C2 = \frac{B3 - B2}{e - c},\ \&c.$$

$$B3 = \frac{A4 - A3}{e - d},\ \&c.$$

&c.

Atque erit Ordinata $T = A +$
$B \times \overline{z-a} +$
$C \times \overline{z-a} \times \overline{z-b} +$
$D \times \overline{z-a} \times \overline{z-b} \times \overline{z-c} +$
$E \times \overline{z-a} \times \overline{z-b} \times \overline{z-c} \times \overline{z-d} +$
$F \times \overline{z-a} \times \overline{z-b} \times \overline{z-c} \times \overline{z-d} \times \overline{z-e} +$
&c.

Notandum est principium Abſciſſæ, nempe punctum R ſumi ad arbitrium, vel inter Ordinatas vel extra omnes ut in Schemate; dummodo ſignorum + & —, debita habeatur ratio. Propoſitio autem demonſtratur ſubſtituendo Ordinatas A, A1, A2, &c. ſucceſſive pro T, & interea pro z longitudines ejus ordine ſuccedentes a, b, c, &c. Nam ſumendo differentias Æquationum provenientium & dividendo eaſdem per intervalla Ordinatarum, obvenient valores Coefficientium ſupra aſſignati.

Exemplum I.

Sint Ordinatarum a principio Abſciſſæ, intervalla $a = 2$, $b = 3$, $c = 5$, $d = 6$; ipſa vero Ordinatæ ſint

A=2,	B=1,	C=0,	$D = \frac{1}{2}$,
A1=3,	B1=1,	C1=2,	
A2=5,	B2=7,		
A3=12,			

Et ineundo computum juxta præſcripta Theorematis, invenietur $B = 1$, $C = 0$, $D = \frac{1}{2}$; quibus ſubſtitutis & 2 pro A, invenietur $T = 2 + \overline{z-2} + \frac{1}{2} \times \overline{z-2 . z-3 . z-5}$, quæ in ordinem reducta fit $T =$

$z^3 - 10$

$\frac{z^3-10z^2+33z-30}{2}$. Etenim si inibi scribantur 2, 3, 5, 6, pro z; prodibunt Ordinatæ propositæ 2, 3, 5, 12.

EXEMPLUM II.

Oporteat determinare tempus Solstitii ex datis aliquot Solis altitudinibus meridianis circa idem tempus. Designent Ordinatæ altitudines solis, earumque intervalla denotent tempora inter observationes; tum transeat Parabola per extremitates Ordinatarum, & Abscissa ejus quæ correspondet minimæ Ordinatæ, sive ea sit una e datis vel quævis intermedia, determinabit momentum temporis quo Sol ingreditur Tropicum. Exempli gratia *B. Walterus* Anno 1500, *Norimbergæ* observavit distantias solis a Zenith ut sequitur

$$\left.\begin{array}{r|l} 44975 & 8^{vo} \\ 44934 & 9^{no} \\ 44883 & 12^{mo} \\ 44990 & 16^{to} \end{array}\right\}\text{Junii}$$

Sit jam distantia observata die octavo Ordinata prima, & in eâdem sit principium Abscissæ; eritque $a=0$, $b=1$, $c=4$, $d=8$; & computus erit

$$\begin{array}{llll} A=44975, & B=-41, & C=+6, & D=+\frac{1}{32}, \\ A1=44934, & B1=-17, & C1=+\frac{175}{28}, & \\ A2=44883, & B2=+\frac{107}{4}, & & \\ A3=44990, & & & \end{array}$$

Atque substituendo hosce valores pro A, B, C, D; invenietur $T=44975-41z+6z.\overline{z-1}+\frac{1}{32}z.\overline{z-1}.\overline{z-4}$, hoc est, $T=44975-\frac{375}{8}z+\frac{187}{32}z^2+\frac{1}{32}z^3$. Jam quoniam Abscissa quæsita correspondet Ordinatæ minimæ, ponatur Fluxio ipsius T æqualis nihil, & habebitur $3zz+374z=1500$, cujus Radix 3.889355 exprimit dies elapsos inter meridiem octavi diei Junii & momentum Solstitii: quod itaque accidit 21 hor. $20\frac{2}{3}$ min. pr. post meridiem diei decimæ primæ, secundum hasce Observationes. Potest etiam tempus Solstitii determinari per plures Observationes & Parabolam plurium dimensionum; vel per tres

Obser-

Obſervationes adhibendo Parabolam Conicam, uti docuit *Halleius.* Sed oportet differentias inter altitudines obſervatas, eſſe ſenſibiliter majores erroribus qui committi queant inter obſervandum, alias nihil certi concludi poterit.

SCHOLION.

Newtonus utitur hac Propoſitione in determinando locum Cometæ qui cadit inter aliquot loca Obſervationibus nota. Nimirum ſi obſerventur quotvis longitudines deſignatæ per totidem Ordinatas, quarum intervalla proportionalia ſunt temporibus inter Obſervationes, & deſcribatur Parabola per Ordinatarum extremitates; Ordinatæ hujus figuræ intermediæ denotabunt longitudines Cometæ intermedias pro temporibus quæ ſunt proportionalia Abſciſſis. Et eadem methodo dabitur latitudo pro quovis tempore ex datis aliquot latitudinibus. Ex longitudine autem & latitudine datis datur locus Cometæ in Cælis. Et hunc in modum plurima obſervatu difficilia determinari poſſint ſatis accurate ex Obſervationibus aliquot anterioribus & poſterioribus.

Applicabilis eſt etiam hæc Propoſitio ad reſolutionem Æquationum purarum affectarumve. Nam in Æquatione reſolvenda ſcribendo pro radice numeros ab eadem haud multum diſcrepantes; provenient eorum intervalla, qui dein interpolati exhibebunt radicem. Sed poſt reſolutionem Æquationum *Hallei* fruſtra compendioſior ſperanda eſt.

In caſu quando intervalla Ordinatarum diminuuntur in infinitum hoc Problema dabit radicem Æquationis fluxionalis, etiamſi nec radix neque altera indeterminata fluat uniformiter; & hoc merâ ſubſtitutione fluxionum Radicis pro differentiis Ordinatarum, & pro earum intervallis fluxiones Abſciſſæ. Nam ſicuti caſus Ordinatarum æquidiſtantium reſpondet fluxionibus Abſciſſæ uniformiter creſcentibus; ita hæc Propoſitio fluxionibus quâcunque lege variantibus. Et reſolutio Æquationis fluxionalis in quâ utraque indeterminata fluit quâcunque lege, non eſt Corollarium ex hac Propoſitione, ſed Caſus ejuſdem omnium ſimpliciſſimus: quod obiter hic monere viſum eſt, ut pateat Methodum Differentialem generaliſſime complecti univerſam Serierum doctrinam, quod forte alii non perceperint.

PROPOSITIO XXX.

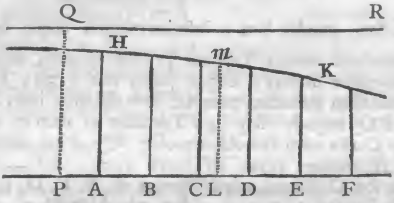
Invenire Aſymptoton Hyperbolæ generis Logarithmici ex datis ejus Ordinatis aliquot æquidiſtantibus.

Sint Ordinatæ æquidiſtantes A, B, C, D, E, &c. inſiſtentes ſuper Abſciſſam PL in angulis rectis: ſitque QR Aſymptotos Curvæ parallela

rallela

rallela Abſciſſæ, & intervallo P Q ab eâdem diſtans. Apelletur vero Abſciſſa quævis A L z, & Ordinata correſpondens L m y. Hyperbola autem Logarithmica H m K definiatur Æquatione hujus formæ

$$y = a - br^{z} - cr^{2z} - dr^{3z} - er^{4z} - \&c.$$



Ubi r, a, b, c, d, e, &c. ſunt quantitates invariabiles; atque erit diſtantia inter Abſciſſam & Aſymptoton, hoc eſt,

$$PQ = A +$$

$$\frac{A - B}{r - 1} +$$

$$\frac{rA - \overline{r+1}B + C}{\overline{r-1}.\overline{r^2-1}} +$$

$$\frac{r^3A - \overline{r^3+r^2+r}B + \overline{r^2+r+1}C - B}{\overline{r-1}.\overline{r^2-1}.\overline{r^3-1}} +$$

$$\frac{r^6A - \overline{r^6+r^5+r^4+r^3}B + \overline{r^5+r^4+2r^3+r^2+r}C - \overline{r^3+r^2+r+1}D + E}{\overline{r-1}.\overline{r^2-1}.\overline{r^3-1}.\overline{r^4-1}} +$$

&c.

Coefficientes literarum A, B, C, D, &c. in diverſis Terminis, formantur per continuam multiplicationem numerorum 1, $\frac{n-1}{r-1}$, $\frac{n-r}{r^2-1}$, $\frac{n-r^2}{r^3-1}$, $\frac{n-r^3}{r^4-1}$, &c. In primo autem Termino eſt $n=r^0$, in ſecundo $n=r^1$, in tertio $n=r^2$, in quarto $n=r^3$, & ſic porro. Exempli gratiâ, in quarto Termino Coefficientes ſumpti ordine inverſo ſunt 1, $1+r+r^2$, $r+r^2+r^3$, r^3, neglectis ſignis; eſt vero $1 \times \frac{r^3-1}{r-1} = 1+r+r^2$, $\overline{1+r+r^2} \times \frac{r^3-r}{r^2-1} = r+r^2+r^3$, denique $\overline{r+r^2+r^3} \times \frac{r^3-r^2}{r^3-1} = r^3$; ſic itaque inveniuntur Coefficientes. In hâc autem Propoſitione excipio

pio casum in quo est $r = \pm 1$; tunc enim degenerat Hyperbola in lineam rectam.

Series vero sic investigatur. In Æquatione assumptâ $y = a - br^z - cr^{2z} -$ &c. mutetur signum Abscissæ z, & prodibit $y = a - \frac{b}{r^z} - \frac{c}{r^{2z}} - \frac{d}{r^{3z}} -$ &c. evadat jam z infinite magna, atque in valore Ordinatæ y evanescent omnes Termini præter primum dummodo concipiatur r esse major unitate; eoque pacto erit $y = a$; hoc est, Ordinata ad distantiam infinitam remota, sive distantia inter Abscissam & Asymptoton PQ æqualis est primo Termino a: nam in distantiâ infinitâ coincidit Curva cum suâ Asymptoto. Quantitas autem a sic investigatur in Æquatione prius assumptâ $y = a - br^z - cr^{2z} - dr^{3z} - er^{4z} -$ &c. scribe Ordinatas æquidistantes A, B, C, D, E, &c. successive pro y, & interea 0, 1, 2, 3, 4, &c. pro Abscissâ z: ac prodibunt Æquationes

$$
\begin{aligned}
A &= a - b - c - d - e - \&c. \\
B &= a - br - cr^2 - dr^3 - er^4 - \&c. \\
C &= a - br^2 - cr^4 - dr^6 - er^8 - \&c. \\
D &= a - br^3 - cr^6 - dr^9 - er^{12} - \&c. \\
E &= a - br^4 - cr^8 - dr^{12} - er^{16} - \&c. \\
&\&c.
\end{aligned}
$$

Habentur igitur tot Æquationes quot sunt incognitæ a, b, c, d, e, &c. ex quibus quæratur a per Algebram vulgarem, atque ejus valor prodibit idem ac jam ipsi PQ assignatus est. Q. E. D.

Corollarium. Hinc si dentur aliquot Termini initiales in Serie infinitâ, quorum differentiæ sunt proxime in progressione Geometricâ, dabitur ultimus omnium utique qui removetur ad distantiam infinitam ab initio. Nam si A, B, C, D, &c. ordine inverso sumpti, designent Terminos quorum differentiæ sunt fere in proportione Geometricâ ut r^0, r^1, r^2, r^3, &c. ultimus omnium æqualis erit PQ intervallo inter Abscissam & Asymptoton: vel stylo *Gregoriano* dabitur Terminatio Seriei.

EXEMPLUM.

Datis Polygonis aliquot regularibus Circulo inscriptis, invenire ultimum Polygonorum sive Aream Circuli. Ea sunto

4	2.00000.00000.0000 = F	3.14033.11569.5475
8	2.82842.71247.4619 = E	126.08888.0294
16	3.06146.74589.2072 = D	6072.7439
32	3.12144.51522.5805 = C	5.5652
64	3.13654.84905.4594 = B	119
128	3.14033.11569.5475 = A	
		3.14159.26535.8979

Dicatur jam ultimum Polygonum A, penultimum B, antepenultimum C, & sic retro. Et quoniam eorum differentiæ A—B, B—C, C—D, &c. sunt quamproxime ut Termini 1, 4, 16, 64, 256, &c. id est, ut dignitates Quaternarii, erit $r=4$; quo substituto, Series generalis evadit

$$A+\frac{A-B}{3}+\frac{4A-5B+C}{3.15}+\frac{64A-84B+21C-D}{3.15.63}+\&c.$$

In qua scribe pro A, B, C, &c. suos valores, & primi quinque Termini dabunt Aream Circuli ad quindecem loca figurarum, ut ex computo apposito manifestum est. Et similiter res consit per Polygona circumscripta.

Hâc autem ratione quævis Series summari potest. Nam si Ordinatæ æquidistantes denotent summas successivas, valor totius Seriei æquabitur distantiæ inter Asymptoton & Abscissam. Si Series summanda sit hujusmodi $a+bx+cx^2+dx^3+$&c. ubi x, x^2, x^3, &c. denotant partes Terminorum quæ sunt in progressione Geometrica, erit $r=x$; & valor ipsius PQ converget eo celerius quo longius Summæ successivæ distant a principio, quæ denotantur per A, B, C, D, &c. In casu autem quando est $r=\pm 1$, assumenda est Hyperbola quæ definitur hujusmodi Æquatione $y=\frac{1}{z^n}$ in $a+\frac{b}{z}+\frac{c}{z^2}+\frac{d}{z^3}+$&c. vice Hyperbolæ Logarithmicæ; atque index n determinabitur ex naturâ Seriei summandæ.

Notandum est Series infinitas æque summari posse per Parabolam *Newtoni* ac per hasce Hyperbolas. Nam Ordinatæ quæ in Hyperbolis sunt æquidistantes, si ad certas quasdam distantias constituantur; exhibebunt, ope Parabolæ, easdem expressiones pro valoribus Serierum.

Numerus figurarum quæ veræ sunt in Polygono A, duplicantur per duos Terminos Seriei, triplicantur per tres, & ita deinceps. Sic in Exemplo allato sunt tres veræ figuræ 314 in Polygono A; & inde quinque Termini dederunt Aream Circuli ad quindecem figurarum loca. Et hæ sunt Approximationes quales olim invenerunt *Jacobus Gregorius* & *Hugenius*: hic quidem triplicavit veras figuras, at ille easdem quadruplicavit, quintuplicavit, imo sine limite produxit, uti videre est in Appendice ad *Veram Circuli & Hyperbolæ Quadraturam*.

PROPOSITIO XXXI.

Invenire Aream cujufvis Curvæ quamproxime ex datis aliquot ejus Ordinatis æquidiftantibus.

Per extremitates Ordinatarum defcribe figuram Parabolicam, ejufque Area quæ invenitur per methodos notas, proxime æquabitur Areæ Curvæ propofitæ. Q. E. I.

SCHOLION.

Quoniam laboriofum effet femper recurrere ad Parabolam, computavi Tabulam fequentem quæ exhibet Aream Curvæ directe ex datis aliquot ejus Ordinatis æquidiftantibus.

Tabula Arearum.

3	$\frac{A+4B}{6}R$
5	$\frac{7A+32B+12C}{90}R$
7	$\frac{41A+216B+27C+272D}{840}R$
9	$\frac{989A+5888B-928C+10496D-4540E}{28350}R$

Tabula Correctionum.

3	$\frac{P-4A+6B}{180}R$
5	$\frac{P-6A+15B-20C}{470}R$
7	$\frac{P-8A+28B-56C+70D}{930}R$
9	$\frac{P-10A+45B-120C+210D-252E}{1600}R$

In hifce Tabulis A eft fumma primæ & ultimæ Ordinatæ, B fecundæ & penultimæ, C tertiæ & antepenultimæ, & fic porro ufque dum deventum

deventum fuerit ad Ordinatam in medio omnium quæ per ultimam literarum A, B, C, &c. repreſentatur. R eſt baſis ſupra quam jacet Area, ſeu pars Abſciſſæ inter primam & ultimam Ordinatam intercepta. P eſt ſumma duarum Ordinatarum quarum una conſiſtit ante primam, altera poſt ultimam ad diſtantias æquales intervallo communi reliquarum Ordinatarum. Numerus autem Ordinatarum qui hic eſt impar, ſignatur ad latera Tabularum. Expreſſiones in Tabula Arearum ſunt Areæ contentæ inter Baſin, Curvam & Ordinatas hinc inde extremas. Eæ vero in Tabula Correctionum ſunt ejuſdem circiter magnitudinis ac differentiæ inter Areas veras & eas per Tabulam prodeuntes: adeoque ſi prima figura Correctionis inveniatur, dein adjiciatur ubi Correctio eſt negativa, vel ſubducatur quando eadem eſt affirmativa; concludere tuto licet Aream ſic correctam, veram eſſe in eo loco decimalium in quo intrat prima figura Correctionis, nec ultra. Itaque per Tabulam Correctionum Area inventa corrigitur, & ſimul numerus figurarum verarum dignoſcitur.

EXEMPLUM.

Sit $\frac{1}{1+x}$ Ordinatæ Hyperbolæ æquilateræ, & quæratur Area ejus quæ jacet ſupra Abſciſſam unitati æqualem. Pro x ſcribe ſucceſſive $\frac{0}{8}, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}$, & provenient novem Ordinatæ

$$\frac{8}{8}, \frac{8}{9}, \frac{8}{10}, \frac{8}{11}, \frac{8}{12}, \frac{8}{13}, \frac{8}{14}, \frac{8}{15}, \frac{8}{16}$$

Igitur eſt $A=\frac{8}{8}+\frac{8}{16}=\frac{3}{2}$, $B=\frac{8}{9}+\frac{8}{15}=\frac{64}{45}$, $C=\frac{8}{10}+\frac{8}{14}=\frac{48}{35}$, $D=\frac{8}{11}+\frac{8}{13}=\frac{192}{143}$, $E=\frac{8}{12}=\frac{2}{3}$: & hiſce ſubſtitutis in ultima expreſſione Arearum, & unitate pro R, oritur Area .69314721. Deinde in Ordinata $\frac{1}{1+x}$ ſcribe $\frac{-1}{8}$ & $+\frac{9}{9}$ ſucceſſive pro x, & provenient duæ Ordinatæ $\frac{8}{7}$ & $\frac{8}{17}$, quarum prior ſtat ante primam, & poſterior poſt ultimam; adeoque $P=\frac{8}{7}+\frac{8}{17}=\frac{192}{119}$: quem ſubſtitue pro P; & pro A, B, C, D, E, ſuos valores, & Correctio pro novem Ordinatis dabit + .00000003, quæ quum affirmativa ſit, ſubducatur a valore Areæ prius invento, & reſtabit .69314718 accurata in ultima figura.

Compu-

Computaveram haſce Tabulas ulterius, ceterum Expreſſiones pro undecem aut pluribus Ordinatis ſunt uſibus inepti propter immenſam magnitudinem Coefficientium numeralium. Sin vero novem Ordinatæ non dent Aream ſatis accuratam; dividatur baſis in duas vel plures partes; & inde Area dividetur in totidem, quarum quamque ſi quæras ſeorſim per novem Ordinatas, habebis totam Aream pro lubitu accuratam. Sed & nonnunquam convenit quærere partem Areæ per Seriem infinitam, præſertim ſi Curva decuſſet baſin in angulo recto. Atque hiſce præcognitis, Area quævis habebitur ſatis accurate per Tabulam jam appoſitam.

Sed & Areæ Curvarum haud incommode exprimi poſſint per Differentias Ordinatarum æquidiſtantium ad modum ſequentem.

Tabula Arearum per Differentias Ordinatarum.

1	A
3	$A+\frac{1}{6}B$
5	$A+\frac{2}{3}B+\frac{7}{90}C$
7	$A+\frac{3}{2}B+\frac{11}{20}C+\frac{41}{840}D$
9	$A+\frac{8}{3}B+\frac{86}{45}C+\frac{92}{189}D+\frac{3}{86}E$
11	$A+\frac{25}{6}B+\frac{175}{36}C+\frac{3445}{1512}D+\frac{4045}{9072}E+\frac{94}{3503}F$
13	$A+6B+\frac{103}{10}C+\frac{158}{21}D+\frac{1833}{700}E+\frac{4813}{11550}F+\frac{66}{3050}G$

In hac Tabula A eſt Ordinata in medio omnium, B eſt differentia ſecunda trium Ordinatarum in medio, C eſt differentia quarta quinque Ordinatarum in medio, & ſic porro ad ultimam literarum A, B, C, D, E, F, G, quæ eſt ultima differentia omnium Ordinatarum. Ut ſi ſint quinque Ordinatæ a, b, c, d, e; erit $A=c$, $B=b-2c+d$, $C=a-4b+6c-4d+e$. Et ſic in aliis caſibus. Expreſſiones autem ductæ in baſin Curvæ, ſive partem Abſciſſæ contentam inter primam & ultimam Ordinatam, dant Areas ex dato numero Ordinatarum qui ſignatur ad latus. Notandum eſt ultimos Terminos in expreſſionibus pro novem, undecem & tredecem Ordinatis, non eſſe veros ſed hiſce ſimpliciores & vero ſatis proximos. Nam Ordinata media A & differentiæ B, C, D, E, &c. conſtituunt Seriem convergentem; ideoque non

non requiritur ut Coefficientes ultimorum Terminorum qui intrant computum, ſint præciſe accurati. Ex convergentia autem Seriei A, B, C, D, &c. dignoſcitur ad quot figuras Area exhibetur accurata; ideoque hæc Tabula non indiget Tabulâ Correctionum. Sed & Coefficientes numerales ſunt longe magis exigui quam in Tabula ſuperiore, atque eâ de cauſâ hæc præferenda eſt, præſertim in magno Ordinatarum numero.

Quæratur rurſus Area Hyperbolæ, cujus Ordinata eſt $\frac{1}{1+x}$; pro x ſcribe o, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$; & provenient quinque Ordinatæ æquidiſtantes 1, $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$, $\frac{1}{2}$: erit igitur $A=\frac{2}{3}$, nempe Ordinatæ mediæ: $B=\frac{4}{105}$, ſcilicet differentiæ ſecundæ trium Ordinatarum in medio $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$: denique $C=\frac{1}{70}$, hoc eſt, æqualis ultimæ differentiæ omnium. Et hi valores ſubſtituti in Expreſſione pro quinque Ordinatis dant $\frac{2}{3}+\frac{8}{315}+\frac{1}{900}=\frac{4367}{6300}$, quæ ducta in baſin ſive unitatem & reducta ad decimales, evadit .69317 pro Areâ quæſita. Et hæc Area ad minimum juſta eſt in quarto decimalium loco, in quo ſcilicet intrat prima figura ultimi Termini $\frac{1}{900}$.

Sed ut differentiæ promptius ac facilius inveniantur; ſit a Ordinata media, b ſumma duarum hinc inde mediæ proximarum, dein c ſumma duarum quæ hinc inde ſequuntur, & ſic porro; tum pone

$$\begin{aligned}
&A=a,\\
&B=b-2A,\\
&C=c-2A-4B,\\
&D=d-2A-9B-6C,\\
&E=e-2A-16B-20C-8D,\\
&F=f-2A-25B-50C-35D-10E,\\
&G=g-2A-36B-105C-112D-54E-12F,
\end{aligned}$$

Atque A, B, C, D, &c. erunt Ordinata media, differentia ſecunda trium mediarum, quarta quinque mediarum, & ſic porro in reliquis reſpective.

PROPOSITIO XXXII.

Designent a, b, c, d, e, f, &c. Terminos æquidistantes continue tendentes ad rationem æqualitatis, & Æquationes sequentes approximabunt ad eorumdem relationes.

2	$a-b=0$
3	$a-2b+c=0$
4	$a-3b+3c-d=0$
5	$a-4b+6c-4d+e=0$
6	$a-5b+10c-10d+5e-f=0$
7	$a-6b+15c-20d+15e-6f+g=0$
8	$a-7b+21c-35d+35e-21f+7g-h=0$
9	$a-8b+28c-56d+70e-56f+28g-8h+i=0$
10	$a-9b+36c-84d+126e-126f+84g-36h+9i-k=0$
	&c.

Hæc Tabula in usum reservanda est, ut consulatur quoties opus sit. Patet autem Coefficientes numerales esse Uncias dignitatum binomii. Et demonstratio patet: quoniam enim Termini supponuntur continue tendere ad rationem æqualitatis, erunt eorum differentiæ $a-b$, $b-c$, $c-d$, $d-e$, &c. parvæ; deinde differentiæ differentiarum $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. erunt minores differentiis primis; atque tertiæ $a-3b+3c-d$, $b-3c+3d-e$, &c. erunt minores secundis; & quartæ $a-4b+6c-4d+e$, &c. erunt minores tertiis, & sic in infinitum. Ergo differentiæ primæ, secundæ, tertiæ, reliquæque positæ æquales nihil ut in Propositione, continue approximabunt ad veram relationem Terminorum. Q. E. D.

Corollarium. Hinc in Serie æquidistantium, si desit Terminus quivis, potest is inveniri per hanc Propositionem. Ut si sint quinque Termini a, b, c, d, e; eorum relatio erit $a-4b+6c-4d+e=0$; & ex hâc Æquatione dabitur eorum quilibet quamproxime ex datis reliquis.

Et notandum est Terminum quemvis, ceteris paribus, eo accuratius definiri, quo propius constet medio omnium: atque errores a vero esse quamproxime ut Coefficientes numerales Terminorum quæsitorum reciproce. Consistat igitur Terminus quæsitus vel in medio omnium vel eidem quantum fieri potest proximus.

EXEMPLUM.

Exemplum.

Quæratur Logarithmus numeri 53 ex datis Logarithmis numerorum aliquot antecedentium. Pone a pro Logarithmo quæſito, eritque

$$\begin{aligned} l,52 &= b = 1.71600.33436 \\ l,51 &= c = 1.70757.01761 \\ l,50 &= d = 1.69897.00043 \\ l,49 &= e = 1.69019.06800 \\ l,48 &= f = 1.68124.12374 \\ l,47 &= g = 1.67209.78579 \end{aligned}$$

Deinde relatio inter ſeptem Terminos a, b, c, d, e, f, g, erit $a - 6b + 15c - 20d + 15e - 6f + g = 0$, adeoque $a = 6b - 15c + 20d - 15e + 6f - g$: ubi ſubſtituendo pro b, c, d, e, f, g, ſuos valores, habebitur 1.72427.58726 pro a ſive Logarithmo numeri 53, exiſtente .00000.00030 errore in exceſſu. Ceterum ſi dentur ſex Logarithmi quorum tres conſiſtunt ante, & tres reliqui poſt illum qui quæritur; in eo inquam caſu Logarithmus quæſitus accuratiſſime definietur.

Igitur deſignent A, B, C, D, &c. ſummas datorum Terminorum qui hinc inde æqualiter diſtant a quæſito, & ejus valores erunt ut in Tabula ſequente

2	$\frac{A}{2}$
4	$\frac{4A - B}{6}$
6	$\frac{15A - 6B + C}{20}$
8	$\frac{56A - 28B + 8C - D}{70}$
10	$\frac{210A - 120B + 45C - 10D + E}{252}$
	&c.

Dentur verbi gratiâ Logarithmi numerorum 50, 51, 52, 54, 55, 56, & oporteat invenire Logarithmum numeri 53. Pone

$$\begin{aligned} l,52 + l,54 &= 3.44839.71034 \\ l,51 + l,55 &= 3.44793.28656 \\ l,50 + l,56 &= 3.44715.80313 \end{aligned}$$

Dein

Dein pro A, B & C ſubſtitue hoſce valores in Expreſſione pro ſex Terminis, & prodibit 1.72427.58695 pro Logarithmo numeri 53, exiſtente unitate errore in ultima figura. Hinc in Tabulis Logarithmicis, Trigonometricis, Aſtronomicis, aliiſque ejuſmodi; ſi deſit forte Terminus quivis, poteſt is inſeri per hanc Propoſitionem; vel ſi ſuſpicio ſit Terminum quemvis erroneum eſſe, poteſt eâdem methodo corrigi. Nam Expreſſiones hic exhibitæ ſunt generales, utique quæ non dependent ex natura Tabulæ cujuſvis particularis.

Propositio XXXIII.

Deſignent &c. $\varepsilon, \delta, \gamma, \beta, \alpha, a, b, c, d, e,$ *&c. Terminos alternos in Serie utrinque excurrente in infinitum, & ponatur* $A=a+\alpha$, $B=b+\beta$, $C=c+\gamma$, $D=d+\delta$, $E=e+\varepsilon$, *& ſic porro; atque Terminus in medio inter* a *&* α *conſiſtens, æqualis erit*

$$\frac{1}{2}\times A +$$
$$\frac{1}{16}\times \overline{A-B} +$$
$$\frac{3}{256}\times \overline{2A-3B+C} +$$
$$\frac{5}{2048}\times \overline{5A-9B+5C-D} +$$
$$\frac{35}{65536}\times \overline{14A-28B+20C-7D+E} +$$
&c.

Coefficientes numerales literarum A, B, C, D, &c. ſunt differentiæ Unciarum in diverſis dignitatibus Binomii: Et Coefficientes qui ducuntur in totos Terminos, ſcilicet $\frac{1}{2}$, $\frac{1}{16}$, $\frac{3}{256}$, &c. generantur per continuam multiplicationem numerorum $\frac{1}{4.2}$, $\frac{3}{4.4}$, $\frac{5}{4.6}$, $\frac{7}{4.8}$, &c. Hiſce vero præcognitis, Series ad libitum producitur.

Et Series inveſtigatur ponendo $z=0$ in Caſu ſecundo Propoſitionis vigeſimæ; tunc enim habebitur Ordinata ſive Terminus qui conſiſtit in medio omnium.

Poſſint

Possint autem Termini Seriei in unam summam colligi ut hic vides,

2	$\frac{A}{2}$
4	$\frac{9A-B}{16}$
6	$\frac{150A-25B+3C}{256}$
8	$\frac{1225A-245B+49C-5D}{2048}$
10	$\frac{39690A-8820B+2268C-405D+35E}{65536}$

Prima Expressio est primus Terminus Seriei, secunda est Summa primi & secundi, tertia est Summa trium primorum Terminorum, & sic deinceps. Ex datis itaque Terminis alternis, intermedii confestim dabuntur per hanc Tabulam vel per ipsam Seriem. Prima Expressio $\frac{A}{2}$ sufficit quando secundus Terminus Seriei minor est quam qui ingrediatur computum. Et similiter in reliquis; nam Termini Seriei sunt differentiæ inter Expressiones & veritatem quamproxime: adeoque semper scire licet quænam expressio sufficit proposito.

Exempli gratiâ, si quæratur Logarithmus numeri 53 ex datis iis numerorum 46, 48, 50, 52, 54, 56, 58, 60; pone

$$l, 52 + l, 54 = A = 3.44839.71035$$
$$l, 50 + l, 56 = B = 3.44715.80313$$
$$l, 48 + l, 58 = C = 3.44466.92309$$
$$l, 46 + l, 60 = D = 3.44090.90820$$

Et hisce valoribus scriptis in Serie, vel in Expressione pro octo Terminis, habebitur 1.72427.58696 pro Logarithmo numeri 53. Et eâdem ratione invenire licet quemvis alium intermedium. In Constructione ergo Tabularum sufficit primo quærere aliquos Terminos in debitis distantiis; nam reliqui possunt hâc methodo inseri. Etenim continue intercalandi sunt Termini primo inventi, usque dum perventum fuerit ad ultimos qui ingrediuntur Tabulam. Et notandum est, oportere omnes Terminos computari circa initium Tabulæ propter magnas differentias; dein per saltum pertransire licet alternos decrescentibus differentiis; & postea ternos, septenosque ubi differentiæ sunt minores. Et hæc est methodus quam præcipit *Newtonus:* ceterum præferendæ sunt regulæ particulares deductæ ex natura Tabulæ construendæ; etenim hæ minore labore plerumque absolvent opus.

F I N I S.

ERRATA.

PAG. 4. l. 5. pro $z+\frac{2}{3}$, lege $z+\frac{3}{2}$.

Pag. 8. l. 18. pro $z+4$, lege $z-4$.

Pag. 44. l. 8. pro $T\times\frac{z-m}{z}$, lege $T\times\frac{z-m}{z}$.

Pag. 67. l. 12. pro $B=\frac{2.3.3}{14.29.29}B$, lege $C=\frac{2.3.3}{14.29.29}B$.

Pag. 76. l. 26. pro $y+xy$, lege $y+x\dot{y}$.

Pag. 80. l. 13. pro $2xyy$, lege $2xy\dot{y}$.

Pag. 83. l. pen. pro eritque, lege erit.

Pag. 91. l. 21. dele &c.

Pag. 94. l. 12. pro $\frac{1}{1999}$, lege $\frac{1}{2000}$.

Pag. 115. l. 12. pro media, lege medio.

Pag. 120. l. 3. pro periferiæ, lege ſemiperiferiæ.

Pag. 122. l. 15. pro 46, lege 4, 6.

Pag. 136. l. 28. pro $x+\frac{1}{2}=10$, lege $x+\frac{1}{2}=11$.

Pag. 137. l. 5. pro $r=\frac{1}{2}$, lege $n=\frac{1}{2}$.

Ibid. l. 21. pro $x=501$, lege $x=501\frac{1}{2}$.

Pag. 147. l. 18. pro Ordinatæ, lege Ordinata.

Ibid. l. 25. pro $\frac{9}{9}$, lege $\frac{9}{8}$.